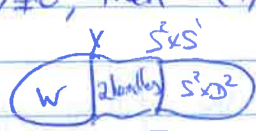
Discussion session: open problems

Q: classify (up to diffeo or symplectic deformation) fillings of any contact manifold not supported by a planar open book (or T^3).
We've seen McDuff and Wendl's theorems; they used some kind of foliation; the index was $n(2-2g)+2c_1(A)$ $\rightsquigarrow$ when genus bigger, the index drops, so it's hard to find foliations.

Theorem: If (Y,ξ) has a filling (W,ω) with $b_2^+(W)>0$ or $b_2^0(W)>0$ or $c_1(W,J)\neq 0$, then (Y,ξ) is not planar. manifolds with boundary can have indefinite part.

Idea.



; find spheres in $S^2\times D^2$. By rational ruled theorem, this is a blow-up of $S^2\times S^2$, so $W \hookrightarrow \#_n\overline{\mathbb{CP}^2}$, so need $b_2^+>0$. "B"

Rem: $\text{ind} = n(2-2g)+2c_1(A) = (n+2)(2-2g)+2A\cdot A$ by adjunction. If we want foliation, $A\cdot A=0$, so the index is negative for g larger than 1.

For many many examples of higher genus OBD, there are infinitely many minimal Stein fillings; could have (Yasi)
- ∞'ly many distinguished by H_*
- ∞'ly many Stein fillings all homeo but not diffeo: [Etnyre-Akhmedov-Smith-Park], [Baykur] by LF, [van Horn Morris]

Rem: there is no homeo class of 4-mflds where we have a classification of the smooth type.

Q: when does (Y,ξ) admit only finitely many fillings?
→ Planar case: are there always ∞'ly many factorizations of monodromy? We know it's true for lens spaces, Seifert fibered spaces, ... But that's pretty much it.
→ Genus 1: have certain types of bounds for fillings $\rightsquigarrow$ hope.
→ Genus ≥ 2: gets out of control. But, see next question.

Q: given (Y,ξ), what is the minimal genus of a supporting OBD?
Can always stabilize to raise genus, but there should be a minimal one.
By Wendl, some things have minimal genus >0 (fillings of planar are neg def and $c_1=0$)
Open question: does there exist (Y,ξ) with min. genus ≥ 2?
Canonical set of examples that people think might work: page = genus g with 1 ∂ component, monodromy = Dehn twist along boundary.

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Rem: no formal obstruction to this: OT mflds admit planar OBD.

Q: is there a homology 3-sphere which is Stein fillable and supported by a planar OBD (other than S^3)?

To "destabilize": check out [Wand]'s stuff. He developped some technology to detect tightness from monodromy. He proved that Legendrian surgery preserves tightness.

Q: can you restrict the topology of the fillings of some contact 3-mfld in any way?
If (Y,ξ) has a contractible Stein filling, does this rule out some possibilities for (Y,ξ)?
Conjecture (Gompf): no Brieskorn sphere has a contractible Stein filling.
Possible approach: find a nice cap, and use some 4-mfld theorem (Donaldson's diagonalization) or symplectic theorem (McDuff's rational ruling)

Q: if (X^4,ω) has $c_1(TX,J)=0$ and $H_*(X;\mathbb{Z})=H_*(K3;\mathbb{Z})$, is it true that $X\cong K3$ diffeomorphism or symplectomorphism?
More generally, are there $(X_1,\omega_1)\not\cong(X_2,\omega_2)$ but $X_1\underset{\text{diffeo}}{\cong}X_2$ sending $c_1(\omega_1)$ to $c_1(\omega_2)$? There are symp def equiv examples without the c_1 condition.

$\exists$ examples of Leg knots with same tb and rot not Leg isotopic; look at Stein mflds obtained from attaching handles to these. See [Etnyre-Honda]'s "Cable of torus knots" or [Tosun].

Q: $\exists$? Liouville cobordism structure on $[0,1]\times Y^3$ which is not $\lambda=e^r\alpha$. Wrong in higher dimensions (McLean): standard ball in exotic ball.
But in dim 4, there is no exotic structures on balls.