Discussion session

Game: Σ surface with boundary, $\mathcal{E} = \{\text{curves in } \Sigma\}$

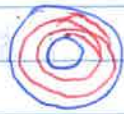
Construction: start with $\Sigma \times D^2$, then attach h_2^4 along the curves in $\mathcal{E}$.

1) $\Sigma =$ , $\mathcal{E} = \emptyset$ $\rightsquigarrow$ get D^4

2) , $\emptyset$ $\rightsquigarrow$ get $T^*S^1 \times D^2 = S^1 \times D^3$

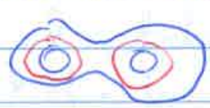
3)  $\rightsquigarrow$ χ: 1 0-h, 1 1-h, 1 2-h $\Rightarrow \chi = 1$

 ; the red curve is  $\Rightarrow$ get $\mathbb{C}^2$

4)  $\rightsquigarrow$ , get T^*S^2

If it goes around 3 times: plumbing $T^*S^2 \# T^*S^2$

5) If 2 1-handles:  or , but they are the same when we cross with D^2

 : get 2 connect sum of , so $\mathbb{C}^2$ again.

 : $\chi = 0$ the red line (2-handle) cancels the left 1-handle.

$\cong$  $=$  $= S^1 \times D^3$

The other way around: (Σ, λ) Liouville ; $(\Sigma \times D^2, \lambda + \lambda_0)$ is Stein.

$$\begin{array}{ccc} \Sigma \times D^2 & \xrightarrow{\text{attach as a critical point}} & X^4 \\ \downarrow & & \downarrow \\ D^2 & & D^2 \end{array}$$

; vanishing cycles correspond to the handles of the critical points

The monodromy around one of the critical values is a Dehn twist.

Theorem: any Stein X^4 admits such a Lefschetz fibration.

Ex: inverse game

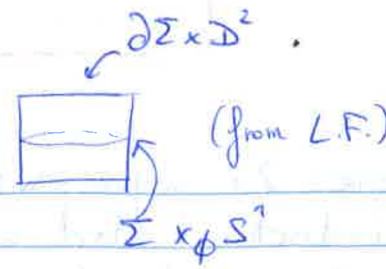
1) T^*S^2:  $T^*S^2 = \{z^2 + w_1^2 + w_2^2 = 1\} \xrightarrow{z}$ (x x)

2) $T^*T^2 = (\mathbb{C}^*)^2_{(x,y)}$ $(x,y) \mapsto x + \frac{1}{x} + y + \frac{1}{y}$, or $x + y + \frac{1}{xy}$.

$\downarrow$ 3-punctured torus

3) $T^*\mathbb{RP}^2$ = complement of projective conic in $\mathbb{CP}^2$

$\rightarrow$ 4-punct. sphere.



Now, it's open books time!
Take (Σ, λ) Liouville, and ϕ a symplectomorphism ($\phi|_{Op(\partial\Sigma)} = id$).
Let's build a contact 3-mfld out of it. Smoothly, get
$\underbrace{\Sigma \times_\phi S^1}_{\lambda + d\theta} \sqcup \underbrace{\partial\Sigma \times D^2}_{ds + r^2 d\theta}$ (s coord. on $\partial\Sigma \simeq S^1$, or a bunch of it)
if no ϕ $\longrightarrow$ for $(-)\times_\phi(\cdot)$: it's ok, can make it work.

Theorem [Giroux] any (Y^3, ξ) has an adapted OB: $(Y^3, \xi) = OB(\Sigma, \phi)$.
(the contact planes will be almost tangent to Σ)

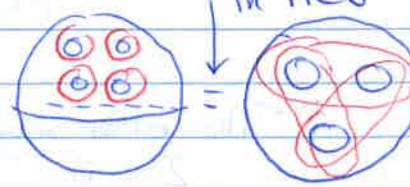
ex: $(Y^3, \xi) = OB(T^*S^1, \tau^2_{S^1})$; we know it is the boundary of the LF with total space T^*S^2 $\rightsquigarrow$ get $\mathbb{RP}^3$.
ex: $(Y^3, \xi) = OB(T^*S^1, \tau^{-1}_{S^1}) = (S^3, \xi)$ (actually, $\xi = \xi_{OT}$)
cf in , tb of the red circle is $0 \to$ OT disk
ex: $\bar\Sigma$ closed, $\omega \in H^2(\bar\Sigma; \mathbb{Z})$, take the bundle with that ~~class~~ Euler class.

$$\begin{array}{ccc} \mathcal{O}(k) & \xrightarrow{\partial} & Y_k \\ \downarrow & & \downarrow S_1 \end{array}$$

Lens space

lantern relations in MCG



$$\begin{array}{ccc} Y^3_1 = S^3 & , & Y^3_k = L(k,1) \\ \downarrow & & \downarrow \\ S^2 & & S^2 \end{array}$$
or $L(-k,1)$

Page = k-punctured sphere, monodromy is positive Dehn twists around each of the punctures
And these are different 4-manifolds, because they have different χ.