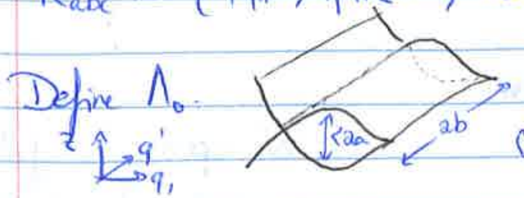
Local model: on $(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i)$, define
$R_{abc} := \{ |q_1|, |p_1| \le 1, |z| \le a, |q'| \le b, |p'| \le c \}$
$q' = (q_2, \dots, q_{n-1})$, $p' = (p_2, \dots, p_{n-1})$

Define Λ_0:  in R_{abc} (in slice $p=0$)

Definition: (R_{abc}, Λ_0) is a standard loose Legendrian chart if $a < bc$.
Definition: a connected Legendrian $\Lambda \subseteq (M^{2n-1}, \xi)$ is loose if $\exists$ Darboux chart $\mathcal{U}$ such that $(\mathcal{U}, \mathcal{U} \cap \Lambda) \cong (R_{abc}, \Lambda_0)$.

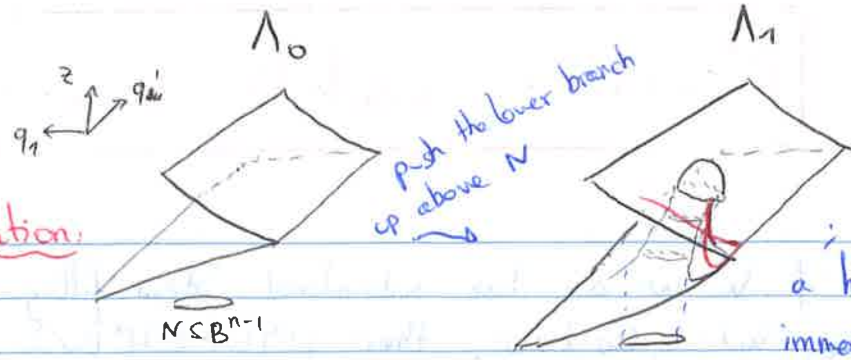
Rem:  , so having a fish is equivalent to having a zigzag.

Rem: the point is also that there is only Λ_0 in that Darboux chart, and not other piece. So, one could have a link composed of a loose Legendrians that it not loose.

Formal Legendrians: $F^s : T\Lambda \to TM$ homotopy of bundle monomorphisms
$\downarrow$
$f : \Lambda^n \hookrightarrow (M^{2n-1}, \xi)$ smooth embedding.
If $F^0 = df$ and F^1 has a Lagrangian image in ξ, then (f, F^s) is a formal Legendrian.
Rem: a Legendrian embedding $f : \Lambda \hookrightarrow M$ is formal: take $F^s = df$ $\forall s \in [0,1]$.
Definition: a formal Legendrian isotopy is a family (f_t, F_t^s) of formal Legendrians.

Existence: given $(f, F^s) \hookrightarrow (M, \xi)$, is it formally isotopic to a Legendrian?
Uniqueness: given (f_t, F_t^s) between f_0, f_1 Legendrian, is there a Legendrian isotopy between them?

Theorem (Murphy) "h-principle for loose Legendrians": $(M^{2n-1 \geq 5}, \xi)$
(E) Given $(f, F^s) \hookrightarrow (M, \xi)$ of Λ^n, $\exists$ loose Legendrian $\hat{f} : \Lambda \hookrightarrow M$ C^0-close to f and formally Leg. isotopic to (f, F^s)
(U) Given (f_t, F_t^s) between $f_0, f_1 : \Lambda \hookrightarrow M$ loose Leg. embeddings, there is a Legendrian isotopy $\hat{f}_t$ from $\hat{f}_0 = f_0$ and $\hat{f}_1 = f_1$, C^0-close to f_t and homotopic to it through formal Leg. isotopies with fixed endpoints.

Stabilization: 

; this is doable as a homotopy through immersed Legendrians.

Proposition (Murphy) * Λ_1 is loose
* $\chi(N) = 0 \Rightarrow \Lambda_1$ is formally leg. isotopic to Λ_0.

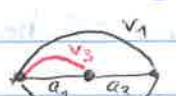
Idea for Λ_1 loose: see the red fish above.

3) Theorem (Cieliebak–Eliashberg) if $\phi \neq \eta$ = homotopy classes of non-degenerate 2-forms on $W^{2n>4}$ (domain, or rel ∂), then
$\mathcal{M}$: $\text{Weinstein}^{flex}_\eta(\omega, X, \phi) \to \text{Morse}_\eta : (\omega, X, \phi) \mapsto \phi$ is surjective, has path-connected fiber, and has the path-lifting property.
Conjecture: $\mathcal{M}$ is a Serre fibration, with contractible fibers.

Theorem (Cieliebak–Eliashberg) "Weinstein h-cobordism": Any flexible Weinstein structure on $W^{2n>4} = Y \times [0,1]$ is homotopic to a Weinstein structure (W, ω, X, ϕ) where $\phi : W \to [0,1]$ has no critical points.
Theorem (Cieliebak–Eliashberg) $(W_i, \omega_i, X_i, \phi_i)$ $(i=1,2)$ two flexible Weinstein structures with $f: W_1 \to W_2$ diffeo st $f^*TW_2 = TW_1$ as sympl. vector bundles, then f is isotopic to a symplectomorphism.
⚠ f compactly supported $\not\Rightarrow$ symplectomorphism compactly supported.

4) Theorem (Casals–Murphy) $X^n_{1,b} = \{(x,y,\underline{z}) \mid xy^b + \sum_{i=1}^{n-1} z_i^2 = 1\} \subseteq \mathbb{C}^{n+1}$ are flexible $\forall\, b \geq 2$.

Theorem: there exists a Weinstein 6-fold (E, λ, ϕ) which is not flexible, but it embeds as a Weinstein sublevel set into the unique flexible Weinstein structure (T^*S^3, λ, ϕ)
Construction: $E = \{(x,y,z,w) \mid x(xy-1) = z^2 + w^2\} \subseteq \mathbb{C}^4$
$\pi \downarrow \quad\quad \downarrow$
$\mathbb{C} \quad\quad x+y$

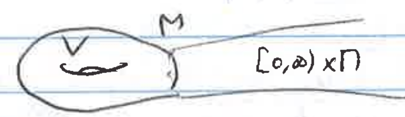


Rem: the h-principle for loose Legendrians is true parametrically, if we fix the loose chart. But it's not written anywhere.

Momchil - Symplectic homology

Goal: if W, W' are two subcritical Stein fillings of (M, ξ) with $c_1(W) = c_1(W') = 0$, then $H^*(W) \cong H^*(W')$.

(V, λ), $d\lambda$ is symplectic, Z st $i_Z d\lambda = \lambda$, and Z is positively transverse to $\partial V \Rightarrow \alpha := \lambda|_{\partial V}$ is contact. Then, $(\hat{V}, \hat{\lambda})$ is the completion $(V, \lambda) \cup_{\partial V} ([0,\infty) \times M, e^r \alpha)$.



Pick $H : \hat{V} \to \mathbb{R}$ and define

$A^H : C^\infty(\mathbb{R}/\mathbb{Z}, \hat{V}) \to \mathbb{R} : x \mapsto \int_{S^1} x^* \hat{\lambda} - \int_{S^1} H(x(t))\, dt$.

Then, $dA^H_x \cdot \xi = \int_{S^1} d\lambda(\xi(t), \dot{x}(t) - X^H(x(t)))\, dt$; for x to be a critical point, need $\dot{x}(t) = X^H(x(t))$.

$\Rightarrow$ crit. points are orbits of H.

Want to do Morse theory. To get a metric on $C^\infty(S^1, \hat{V})$, put one on $\hat{V}$ first: pick $J : T\hat{V} \to T\hat{V}$, $J^2 = -1$, $d\lambda(-, J-)$ is a Riemannian metric. For this, $\nabla_x A^H = -J(\dot{x} - X^H(x))$.

The upward gradient flow is: for a path $u : \mathbb{R}_s \times S^1 \to \hat{V}$, have $\partial_s u + J(\partial_t u - X^H) = 0$ "Floer's equation"

$\rightsquigarrow$ "Just do Morse theory": let $CF(H) := \bigoplus_{x \in \mathrm{crit}(A^H)} \mathbb{F}_2 \cdot x$

For $c_1(V) = 0$, $CZ(x) \in \mathbb{Z}$; define $|x| = CZ(x) + n - 3$. Define a differential by $d : CF_k(H) \to CF_{k-1}(H)$:

$$d(x_+) = \sum_{\substack{x \in \mathrm{crit}(A^H) \\ |x_-| = |x_+| - 1}} \#_{\mathbb{F}_2} \underbrace{M(x_-, x_+)}\, x_-$$

$$\hookrightarrow \left\{ u : \begin{array}{l} \partial_s u + J(\partial_t u - X^H) = 0, \\ \lim_{s \to \pm\infty} u(s,t) = x_\pm(t) \end{array} \right\} \Big/ \mathbb{R} \quad \mathbb{R}\text{-translation}$$

For these indices, that is 0-dimensional, but we don't know whether it's finite.

Compactness: Gromov compactness works if the following holds:

(a) uniform C^0 bounds

(b) uniform energy bounds.
$\hookrightarrow E(u) = A^H(x_+) - A^H(x_-)$
$\Rightarrow d$ decreases action.

(b) comes for free, as $A^H(x_+)$ and $A^H(x_-)$ are fixed.
(a) needs some serious restrictions.

write $h(t) = At + B$

Definition: the spectrum $\mathrm{Spec}(\Gamma, \alpha) := \{T \in \mathbb{R} \mid \exists$ Reeb orbit of α with period $T\}$.

Definition: the space of admissible Hamiltonians is

$Ad(V, \lambda) := \{ H : \hat{V} \to \mathbb{R} \mid$ ~~outside a compact set, $Z(H) = H$ and $\mathrm{slope}(H) \notin \mathrm{Spec}(\Gamma,\alpha)$~~

~~Rem. $Z = \partial/\partial r$, so this means~~ $H: \hat{V} \to \mathbb{R}$ looks like $H(r,y) = Ae^r + b$ in the cylindrical end, and $A \notin \mathrm{Spec}(\Gamma, \alpha)$ is ~~not in~~ the slope. $\}$

Definition: a $J : T\hat{V} \to T\hat{V}$ is called cylindrical if on the cylindrical end, $J = j \oplus J_\xi$, st $j\partial_r = R$ and $J_\xi : \xi \to \xi$ is a $\mathbb{C}$-str. on ξ.

These choices ensures C^0 bounds $\Rightarrow$ compactness $\Rightarrow$ d is well-defined and $d^2 = 0$.

Definition. $HF_k(H) := H_k(CF_*(H), d)$.

Rem: a priori this should depend on J, but in fact, it does not.

In the cylindrical end, if $u(s,t) = (a(s,t), v(s,t))$, the equation is $\Delta a + \partial_s(h'(e^a)) = \|\partial_s v\|^2 \geq 0$, so by maximum principle, we can't have max in interior:

if there is one, have $\begin{cases} \Delta a \leq 0 \text{ because max} \\ \partial_s(h'(e^a)) = h''(e^a)e^a \underbrace{\partial_s a}_{=0} \leq 0 \text{ because max,} \end{cases}$

hence contradiction. (or almost, since the whole thing could be $= 0$).

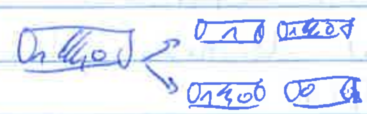
Independence of H?

$H_+, H_- \in Ad(V, \lambda)$. Let $H_s : \hat{V} \to \mathbb{R}$ path from $\underset{H_{s \ll 0}}{H_-}$ to $\underset{H_{s \gg 0}}{H_+}$, and interpolate by considering solutions to the equation

$$\partial_s u + J(\partial_t u - X^{H_s}(u)) = 0 \qquad \text{⚠ no } \mathbb{R}\text{-translation.}$$

Counts

Define $\Phi : CF(H_+) \to CF(H_-)$, $\Phi(x_+) = \sum_{\substack{x_- \in \mathrm{Crit}(A^{H_-}) \\ |x_+| = |x_-|}} \# \mathcal{M}(x_-, x_+)\, x_-$

Possible breakings:  so it's a chain map

For compactness, we get the equation $\Delta a + \partial_s(h_s'(e^a)) = \|\partial_s v\|^2 \geq 0$, but there is a $(\partial_s h_s')(e^a)$ is there, that might ruin the previous argument. However, it's fine if $\partial_s h_s' < 0$, i.e. if H_- has steeper slope than H_+.

So: get only a map in one direction.

$\rightsquigarrow$ Define a partial order on $Ad(V, \lambda)$: we say $H_1 \ll H_2$ if $H_1 < H_2$ outside a compact set (ie $\neq$ slopes, or one is shift of the other).

Also, by a homotopy of homotopies argument,

$$\begin{tikzcd} HF(H_1) \arrow[r] \arrow[rr, bend right] & HF(H_2) \arrow[r] & HF(H_3) \end{tikzcd}$$ commutes.

Also, this forms a directed system:

$$\begin{tikzcd} HF(H) \arrow[dr] & \\ & HF(\tilde{H}) \\ HF(H') \arrow[ur] & \end{tikzcd}$$

Definition. $SH_k(V) = \varinjlim_{H \in \mathcal{H}(V,\lambda)} HF_k(V)$; can compute it using a cofinal sequence.

→ What are the generators? Take a cofinal sequence of Hamiltonians looking like this, with increasing slope in the cylindrical end.

Fact 1: if H is C^2-small ($\|\mathrm{Hess}\, H\| \ll \varepsilon$), then all the 1-periodic orbits of X^H are constant, ie crit pts of H. Moreover, the Floer trajectories connecting crit pts of H are just Morse flow lines.

Compute: $X^H = h'(e^r)R$, so orbits lie in constant r level, and they coincide with Reeb orbits, up to reparametrization, of period $h'(e^{r_0}) = T$, and has action $A^H(x) = e^{r_0}T - h(e^{r_0})$
$\geq T - h(e^{r_0}) > 0$

~~There is~~ So, due to the hypothesis on slope & spectrum, we see that there are no Reeb orbits in the linear part.

Also, there is $0 \to \overbrace{CF^{<0}(H)}^{\text{crit pts}} \to CF(H) \to \overbrace{CF(H)/CF^{<0}(H)}^{\text{Reeb orbits}} \to 0$,

because the differential decreases action. So, get exact triangle

$$\begin{tikzcd} SH^{<0}(V) \text{ "="} H^{-k}(V) \arrow[r] & SH_k(V) \arrow[dl] \\ SH^+_k(V) \arrow[u] & \end{tikzcd}$$

[Bourgeois-Oancea]

Theorem: if all Reeb orbits of (Π, α) satisfy $CZ(\gamma) + n - 3 > 0$ and V, W are 2 exact fillings of Π with $c_1(V) = c_1(W) = 0$, then $SH^+(V) \simeq SH^+(W)$. So, it's an invariant of the compact manifold.

Proof: we show no cylinder goes into the filling. Stretch the neck: if we have a seq. of cylinders going in the filling, stretch the neck ⇒ it has to break in or , ruled out by index: it will be negative. □

Theorem (M-L Yau) if M is subcritically Stein fillable with $c_1(\text{filling}) = 0$, then M admits a contact form st $CZ(\gamma) + n - 3 > 0$ $\forall$ Reeb orbit γ.

Theorem. (Cieliebak) V subcritical Stein with $c_1(V) = 0 \Rightarrow SH(V) = 0$.

So, we can prove the goal: $SH(V) = SH(W) = 0$, and $SH^+(V) = SH^+(W)$, so by the exact triangle above, we have $H^*(V) = H^*(W)$. The index assumption on Reeb orbits is satisfied by Yau's theorem.

Rem: Floer cylinders have to approach the negative orbit "by the right a little bit": $\exists\, (s_0, t_0) \in \mathbb{R} \times S^1$ st $a(s_0, t_0) > r_-$, where r_- is the r-level of the negative Reeb orbit. This is due to Bourgeois–Oancea, and helps to rule out  above: must have  for the bottom right one, but that contradicts the maximum principle.
One should remember this as one of the fundamental properties of holomorphic curves, along with the maximum principle. (Kyler).

Cédric - Computations on Brieskorn manifolds

Goal: use SH^+ to distinguish contact structures

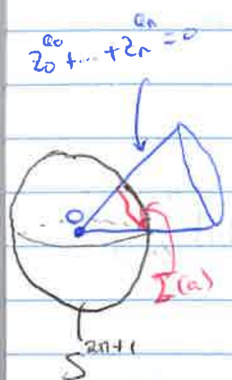
Plan: 1) Brieskorn manifolds

2) Ustilovsky's exotic spheres

♪ Interlude ♪ Morse-Bott things

3) Uebele's computation of SH^+ for fillings of $\Sigma(2l,2,\dots,2)$.

1) Brieskorn manifolds:



Definition: $\Sigma^{2n-1}(a_0,\dots,a_n) := \{z\in\mathbb{C}^{n+1} \mid z_0^{a_0}+\dots+z_n^{a_n}=0\}\cap S^{2n+1}\subseteq \mathbb{C}^{n+1}$.

Write $a=(a_0,\dots,a_n)$ and $\Sigma(a)$ for $\Sigma(a_0,\dots,a_n)$.

→ Contact form on $\Sigma(a)$: $\alpha_a = \frac{i}{8}\sum_{j=0}^{n} a_j(z_j\,d\bar z_j - \bar z_j\,dz_j)$.

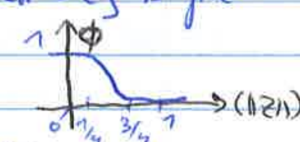
Rem: $\mathbb{C}^{n+1}$ admits an a-weighted Hermitian form: $\langle\xi,\zeta\rangle_a = \frac{1}{2}\sum_{k=0}^{n} a_k \xi_k \bar\zeta_k$, with associated (wrt i) symplectic form $\omega_a = -\mathrm{Im}\langle\cdot,\cdot\rangle_a = \frac{i}{4}\sum a_k\,dz_k\wedge d\bar z_k$.

Then $\frac{z}{2}$ is Liouville for ω_a, with Liouville 1-form $\lambda_a = \omega_a(\frac{z}{2},-) = \frac{i}{8}\sum a_k(z_k\,d\bar z_k - \bar z_k\,dz_k)$.

→ Reeb vector field: $R_{\alpha_a} = \left(\frac{4i}{a_0}z_0,\dots,\frac{4i}{a_n}z_n\right)$

→ Reeb flow: $\phi_t^a(z) = (e^{4it/a_0}z_0,\dots,e^{4it/a_n}z_n)$

→ Filling: we'd like to fill $\Sigma(a)$ by $\{z\in\mathbb{C}^{n+1}\mid z_0^{a_0}+\dots+z_n^{a_n}=0\}$, but this has a singularity at 0. However, $\{z\in\mathbb{C}^{n+1}\mid z_0^{a_0}+\dots+z_n^{a_n}=\varepsilon\}$ is fine. we take an interpolation: $W_a := \{z_0^{a_0}+\dots+z_n^{a_n} = \varepsilon\,\phi(\|z\|)\}$. For ε small, this is an exact filling that is parallelizable, hence $c_1(W_a)=0$. Also, it has the homotopy type of $\bigvee_{\mu(a)} S^n$ (wedge product), where $\mu(a)=\prod_{i=0}^{n}(a_i-1)$.



Theorem: 1) $\pi_1(\Sigma(a))=\dots=\pi_{n-2}(\Sigma(a))=0$, and there is an algorithm ([Randell]) to compute $H_{n-1}(\Sigma(a))$.

$n\geq 3$ 2) $\Sigma(a)$ is homeomorphic to S^{2n-1} $\iff$ $\exists\, a_i,a_j$ which are relatively prime to all other exponents, OR $\exists\, a_i$ which is relatively prime to all the other exponents, and a set $\{a_{j_1},\dots,a_{j_r}\}$ ($r\geq 3$ odd) such that each a_{j_k} is relatively prime to any exponent not in the set, while $\gcd(a_{j_k},a_{j_\ell})=2$ $\forall k\neq\ell$.

Fun facts: 1) $\Sigma(2,2,2,3,6k-1)$ for $k=1,\dots,28$ give all 28 smooth structures on the oriented S^7.

2) Any M^5 spin with $\pi_1(M)=0$ is a connect sum of $\Sigma(a)$'s.

2) Ustilovsky's exotic spheres

Let $(\Sigma_p^m, \xi_p^m) := (\Sigma(p, \overbrace{2, \dots, 2}^{2m+1}), \ker \alpha_{(p,2,\dots,2)})$, with an odd number of 2's.
By the criterion above, these manifolds are homeomorphic to spheres.
[Brieskorn]: for $p \equiv \pm 1 \mod 8$, Σ_p^m is diffeomorphic to S^{4m+1} standard.
Recall: $\alpha_p^m = \frac{ip}{8}(z_0 d\bar{z}_0 - \bar{z}_0 dz_0) + \frac{i}{4}\sum_{j=1}^{2m+1}(z_j d\bar{z}_j - \bar{z}_j dz_j)$, $\xi_p^m := \ker(\alpha_p^m)$

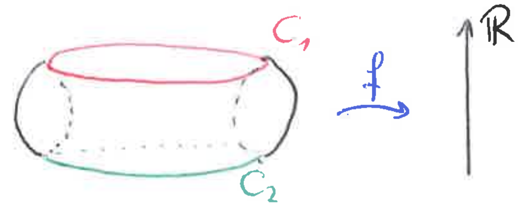
Theorem (Ustilovsky) $p_1 \neq p_2 \Rightarrow \xi_{p_1}^m$ is not contactomorphic to $\xi_{p_2}^m$.
"Proof": use contact homology. Take an explicit perturbation of α_p^m so that all the Reeb orbits are non-degenerate. Compute their degrees; find that they are all even. Since the degree of the differential is -1, it vanishes, hence contact homology is isomorphic to the contact algebra. For different values of p, the degrees of the generators differ, hence contact homologies are not isomorphic. But contact homology is an invariant of the contact structure. □

Now, we will see that infinitely many of those are "homotopically equal".
Definition: an almost contact structure on Y^{2n+1} is a pair (α, β) where α is a 1-form and β a 2-form, such that $\alpha \wedge \beta^n$ is non-vanishing. This is the same as a reduction of the structure group of TY to $U(n) \times 1$.
ex: a contact structure $\xi = \ker \alpha$ gives the almost contact structure $(\alpha, d\alpha)$.
The homotopy class among almost contact structures of a contact structure is called its formal class.
Definitions: a contact structure on S^{2n+1} is
* exotic if it is not contactomorphic to (S^{2n+1}, ξ_{std})
* homotopically trivial if it is in the formal class of (S^{2n+1}, ξ_{std})

An almost contact structure on S^{4m+1} $\longleftrightarrow$ lift
$$\begin{tikzcd} & B(U(2m)\times 1) \arrow[d] \\ S^{4m+1} \arrow[r] \arrow[ur, dotted] & BSO(4m+1) \end{tikzcd}$$
and those are classified by $G := \pi_{4m+1}(SO(4m+1)/U(2m)\times 1)$, since $SO(4m+1)/U(2m)\times 1$ is the fiber of the vertical map.
[Massey]: G is cyclic of order $d = \begin{cases} (2m)! & \text{if } m \text{ is even} \\ \frac{(2m)!}{2} & \text{if } m \text{ is odd} \end{cases}$
[Morita]: $[\xi_p^m]$ above is $\frac{p-1}{2}$ mod d in that group.
$\Rightarrow$ for $p \equiv 1 \mod 2\cdot(2m)!$, ξ_p^m is homotopically standard. So we get:

Theorem: $\exists$ infinitely many homotopically trivial exotic contact structures on S^{4m+1}.
Proof: for fixed m, $\exists$ infinitely many p st $p \equiv \pm 1 \mod 8$ and $p \equiv 1 \mod 2\cdot(2m)!$ □

ex:



Interlude: Morse-Bott things

In finite dimension, how to do Morse homology if the critical points of $f: M \to \mathbb{R}$ are not isolated?

Definition: $f: M \to \mathbb{R}$ is Morse-Bott if $\mathrm{crit}(f) = \bigsqcup C_i$ is a disjoint union of connected submanifolds, such that $\mathrm{Hess}_p f|_{\nu(C_i)}$ is non-degenerate $\forall p \in C_i$. ex: see above.

What we could do to compute Morse-homology is choose h a Morse function on $\mathrm{crit}(f)$; then $f + \varepsilon \rho h$ is Morse (for ε small and ρ an appropriately chose cutoff near $\mathrm{crit}(f)$), and its critical points are exactly those of h. But we'd rather not break the symmetry, for the sake of computability.

Idea: crit f $\overset{\varepsilon \to 0}{\rightsquigarrow}$ crit f, so maybe counting these "cascades" could work...
grad flow of $f + \varepsilon \rho h$ — grad flow of f — grad flow of h

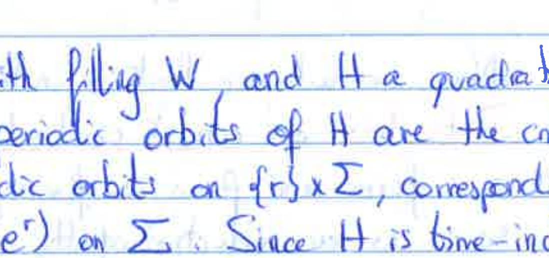
$\hookrightarrow$ For SH^+: let (Σ, ξ) contact with filling W and H a quadratic Hamiltonian, C^2-small in W. The periodic orbits of H are the critical points of H in W, and the 1-periodic orbits on $\{r\} \times \Sigma$, corresponding to the closed Reeb orbits of period $h'(e^r)$ on Σ. Since H is time-independent, these come at least in S^1-families.

[Bourgeois-Oancea]: $SH_{\text{Morse-Bott}}$ if orbits come just in S^1-families. But it also works here, under the following Morse-Bott condition:
$N_T = \{z \in \Sigma \mid \phi_T(z) = z\}$ is a closed submanifold, such that $\mathrm{rk}(d\alpha|_{N_T})$ is locally constant and $T_p N_T = \ker(d\phi_T - \mathrm{id})$. To have grading, assume that $c_1(W)$ and that the closed Reeb orbits are contractible in Σ.

Choose a Morse function f_T on each N_T.

$\to$ Generators of $SC^+(W)$: (T, η) where $\eta \in \mathrm{crit}(f_T)$

$\to$ Differential: Counts isolated trajectories of the form in the picture, where the lines "A^H" denote Floer cylinders.



(*) $\to$ Grading: $\mu(T, \eta) = \mu_{CZ}(N_T) + \mathrm{ind}_{f_T}(\eta) - \frac{1}{2}(\dim N_T - 1)$.

Why? 1) If we perturb $f + \varepsilon \rho h$, that is what we get.
2) With this grading, the differential has degree -1.

End of interlude

3) Uebele's computations:

We now focus on $\Sigma_\ell^n = \Sigma(2\ell, \underbrace{2, \dots, 2}_{n \text{ odd}})$. This has $R_\alpha = 2i\left(\frac{z_0}{\ell}, z_1, \dots, z_n\right)$ and $\phi_t(z) = \left(e^{2it/\ell} z_0, e^{2it} z_1, \dots, e^{2it} z_n\right)$, by the formulae in section 1).

→ Randell's algorithm gives $H_{n-1}(\Sigma_\ell^n) \simeq \mathbb{Z}$.

→ Wall's classification of highly-connected mflds: $\Sigma_\ell^n \underset{\text{diffeo}}{\cong} \begin{cases} S^{n-1} \times S^n & \text{if } \ell \equiv 0 \bmod 4 \\ S^* S^n & \ell \equiv 1 \bmod 4 \\ (S^{n-1} \times S^n) \# K & \ell \equiv 2 \bmod 4 \\ S^* S^n \# K & \ell \equiv 3 \bmod 4 \end{cases}$,

where $K = \Sigma(3, 2, \dots, 2)$ is the Kervaire sphere.

Rem: if $n = 3$, all give $S^2 \times S^3$, as $K \cong S^5_{\text{std}}$.

Theorem: the manifolds Σ_ℓ^n are pairwise non-contactomorphic.

To prove this, we compute SH^+(standard filling), and use the following lemma:

Lemma: for Σ_ℓ^n, the SH^+ is independent of the chosen filling W, as long as $c_1(W)|_{\pi_2(W)} = 0$.

So, let us compute SH^+, using the Morse-Bott setup presented above.

1) Critical manifolds: ϕ is periodic everywhere: if $z_0 = 0$ then period π, and if $z_0 \neq 0$, then period $\ell\pi$. So, we get: $N_T = \begin{cases} \Sigma_\ell^n & \text{if } T = N\pi, \ell \mid N \\ \Sigma_\ell^n \cap \{z_0 = 0\} & \text{" }, \ell \nmid N \\ \emptyset & \text{else.} \end{cases}$

2) Morse functions on N_T:

* $\Sigma_\ell^n \cap \{z_0 = 0\} \simeq S^* S^{n-1} \Rightarrow \exists$ perfect Morse fct, with indices $\in \{0, n-2, n-1, 2n-3\}$
* $\Sigma_\ell \Rightarrow$ [Fauck]: can pretend " " " " " " $\in \{0, n-1, n, 2n-1\}$

3) To compute Robbin-Salamon indices: by additivity of μ_{CZ}, we do it on $T\mathbb{C}^{n+1}$ and on $\xi_\ell^\perp$, which is generated by 4 explicit vector fields. For $N_{N\pi}$, get:

→ On $T\mathbb{C}^{n+1}$: $\Phi^t = D\phi^t = \operatorname{diag}\left(e^{\frac{4it}{2\ell}}, \dots, e^{\frac{4it}{2}}\right) \rightsquigarrow \mu_{CZ}(\Phi) = \sum_{k=0}^{n} \left(\left\lfloor \frac{4 \cdot N\pi}{2\pi a_k} \right\rfloor + \left\lceil \frac{4N\pi}{2\pi a_k} \right\rceil \right)$

→ On $\xi_\ell^\perp$: $\Phi_2^t = \operatorname{diag}(e^{4it}, 1)$ for some explicit basis of $\xi_\ell^\perp$.

$\Rightarrow \mu_{CZ}(\gamma \in N_{N\pi}) = \begin{cases} 2N/\ell + 2N(n-2) & \text{if } \ell \mid N \\ 2\lfloor N/\ell \rfloor + 2N(n-2) + 1 & \text{if } \ell \nmid N \end{cases}$

4) Degrees of generators for SC^+: use the formula (*) above

→ $\ell \mid N$: get $2\frac{N}{\ell} + 2N(n-2) + \{-n+1, 0, 1, n\}$

→ $\ell \nmid N$: get $2\lfloor\frac{N}{\ell}\rfloor + 2N(n-2) + \{-n+3, 1, 2, n\}$

5) The differential: within a critical manifold, the Morse function we chose is perfect, so the differential vanishes. From 1 mfld to the other: if $n \geq 5$, can see that $\exists!$ 1 pair of generators for each consecutive degrees, achieved by: ind(gen. with period $N+1$) = ind(gen. with period N) $- 1$. But ∂ decreases the period, hence $\partial = 0$. For $n = 3$, can use a symmetry to prove that a lot of the differentials still vanish.

$\Rightarrow$ So $SH^+ = SC^+$ (or close to it if $n=3$), and by looking at the degrees, we see they differ for different values of ℓ. $\square$

Scott - Contact manifolds with flexible fillings

Ref: [Lazarev]

Let (Y^{2n-1}, ξ) contact, W^{2n} Weinstein filling $\rightsquigarrow$ have handles $\leq n$.

Theorem (Yau) if W_1, W_2 are 2 subcritical fillings, then $H^*(W_1) = H^*(W_2)$

Theorem 1 [Lazarev] if W_1, W_2 are flexible fillings of (Y^{2n-1}, ξ), then $H^*(W_1; \mathbb{Z}) \cong H^*(W_2; \mathbb{Z})$.

Rem: if Y has 1 flexible filling, then replacing W_2 by any Liouville filling with $SH = 0$ gives the same conclusion.

Theorem 2, if $n \geq 3$ and (Y^{2n-1}, ξ) has a flexible filling, then $\exists$ infinitely many (ξ_M) such that $\xi_M \not\cong \xi$, such that (Y^{2n-1}, ξ_M) has flexible fillings.

Proof: of theorem 1 $\Rightarrow$ theorem 2: (Y, ξ) $\cap\!\!\!|$ W ; (S^{2n-1}, ξ) $\cap\!\!\!|$ as flexible fillings $\exists M$

Take a boundary connect sum (ie attach 1-handle):

$\rightsquigarrow (Y, \xi) \# (S^{2n-1}, \xi) \simeq (Y, \xi)$

$\cap\!\!\!|$ as f.f.

filling of $W \natural M$ (flexible)

And $H^*(W \natural M) = H^*(W) \oplus H^*(M)$ so we can make their homology different. How to get these M?

$\rightarrow$ n odd: take M = Brieskorn mfld, $\dim H^n(M_i) \to \infty$.

$\rightarrow$ n even: [Geiges] "Applications of contact surgery" to find filling of spheres with $\dim H^n(M) \geq 1$ $\rightsquigarrow$ take $M_i = \natural_i M$. $\square$

Rem: $(Y, \xi) \overset{\text{homotopic}}{\sim} (Y, \xi_M)$.

Now, let' walk towards proving theorem 1.

Definition: (Y^{2n-1}, ξ) is dynamically convex (DC) if $\exists \alpha$ such that all Reeb orbits have positive degree: $|\gamma| = \mu_{CZ}(\gamma) + n - 3 > 0$.

Theorem: if DC, then SH^+ is indep. of the Stein filling, if $c_1 = 0$.

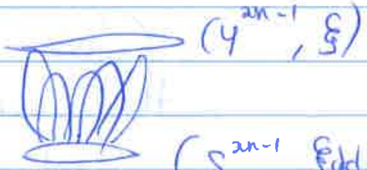
Definition: (Y^{2n-1}, ξ) is asymptotically DC (ADC) if $\exists\ \alpha_1 \geq \alpha_2 \geq \dots$ (contact forms) and $D_1 < D_2 < \dots \to \infty$ such that $P^{<D_i}(Y, \alpha_i)$ (set of Reeb orbits of action $<D_i$) all have positive degree.

Theorem. if ADC, then SH^+ is indep. of the Stein filling, if $c_1 = 0$.

So: we want to show that flexible fillings are ADC: $\begin{array}{ccc} H & \longrightarrow & SH = 0 \\ \uparrow & \swarrow & \\ SH^+ & & \end{array}$, so $SH^+ \simeq H$, and SH^+ does not depend on W_1 or W_2, and hence $H^*(W_1) \simeq H^*(W_2)$.

Rem: $\alpha_1 \geq \alpha_2 \geq \dots$ is important; without that, being ADC would be trivial. ~~Since~~; This is a continuation map situation; if $\alpha_1 \geq \alpha_2$, can connect them by a piece of symplectization.

If (Y^{2n-1}, ξ) has flexible filling W^{2n}:



Theorem: if $(Y_-, \xi) \to (Y_+, \xi)$ through subcritical (Yau) or flexible (Oleg) surgery, then Y_- ADC $\Rightarrow Y_+$ ADC.

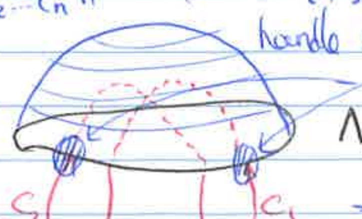
Since (S^{2n-1}, ξ_{std}) is ADC, this would prove that being flexible is ADC, hence SH^+ indep of filling by theorem above. So, we just have to prove the theorem.

Rem: when you do surgery, that ~~deg~~ decreases the contact form. Being ADC is important; it wouldn't work for just DC.

Proposition: [Bourgeois - Ekholm - Eliashberg] after surgery along an attaching Legendrian sphere Λ^{n-1} $(n \geq 3)$, we have

* $\begin{Bmatrix} \text{new Reeb orbits} \\ \text{with period} < D \end{Bmatrix} \overset{1-1}{\longleftrightarrow} \begin{Bmatrix} \text{cyclic words of Reeb chords} \\ \text{in } Y \text{ on } \Lambda \text{ with period} < D \end{Bmatrix}$
* $\| \gamma_{c_1 c_2 \dots c_n} \| = \left(\sum \| c_i \| \right) + n - 3$

"Proof":



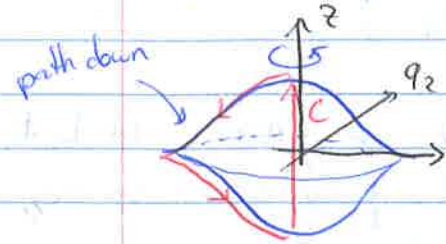
Given c_1, c_2: perturb so that end of c_1 flows to beginning of c_2 in the handle, and end of c_2 to beginning of c_1. (c₁ and c₂)
$\Rightarrow$ The resulting orbit is close to c_1 and c_2.

Conversely, an orbit clearly gives a word. $\square$

Rem: the Reeb flow looks like the geodesic flow on the disk. In particular, all orbits leave the disk.

Key lemma: if $\Lambda^{n-1} \subseteq Y$ $(n \geq 3)$ is loose, $\exists$ Legendrian isotopy such that (period-bounded) Reeb chords have positive degree.

In $(\mathbb{R}^{2n-1}, dz - \sum p_i dq_i)$, ex $S^2 \subseteq \mathbb{R}^5$, consider



R_α is $\partial/\partial z$, so we want vertical lines in the front proj. between points with the same slope.

Given c a Reeb chord, follow it, and then choose a path that comes back down:

$|c| = \underset{\#\text{down cusps}}{D} - \underset{\#\text{up cusps}}{U} + \text{ind}_{h_2 - h_1}(p) - 1$, where h_1 and h_2 are the functions giving z in terms of q_i's.

$= 1 - 0 + 2 - 1 = 2$

To increase number of downward cusps: stabilize

But we introduce new Reeb chords (red). For them:

$|c| = 2 - 0 + (\text{ind} \geq 0) - 1 \geq 1$, so it's ok.

Also, since Λ is loose, one can realize this stabilization by an isotopy.

Rem: we have this decreasing sequence of contact forms because of this stabilization, and because ~~attach~~ doing surgery also tends to make the handles thinner.

Discussion session

Q: Liouville vs Weinstein.
Weinstein $\Leftrightarrow$ $\frac{1}{2}$ dim'l homotopy type
ex: (McDuff) There is a Liouville structure on (W, λ), where
$W \underset{C^\infty}{\cong} \mathcal{U}^* \Sigma_g$ $\rightsquigarrow$ homotopy type of 3D mfld, so it can not be Weinstein.
What about $W \times \mathbb{C}$? This could be Weinstein...
Q/exercise: find by hand a Liouville structure on a 3-handle.