Kylerec 2017

Symplectic fillings of contact manifolds

Truckee CA, May 19th - 25th 2017

Tuesday May 23rd, 2017:

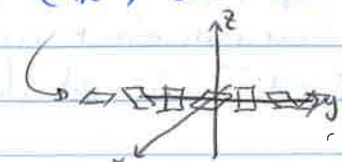
Wednesday May 24th, 2017:

Notes taken by Cédric De Groote (cedricd@stanford.edu)
The workshop took place from Friday May 19th, 2017 to Thursday May 25th, 2017 at 14742 Skislope Way, Truckee, CA 96161.

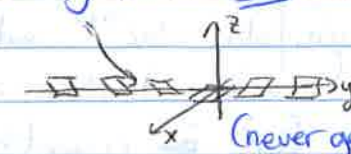
Roger, Laura - Introduction, types of fillings / fillability

From contact viewpoint:

[Bennequin, 1982]: proves that $(\mathbb{R}^3, dz - y\,dx)$ is not contactomorphic $\cong (\mathbb{R}^3, dz + r^2 d\theta)$ to $(\mathbb{R}^3, \cos r\, dz + r \sin r\, d\theta)$.



(keeps turning)



(never gets vertical)

Rem: $\exists$ diffeo of $\mathbb{R}^3$, taking $dz - y\,dx$ and making it twist in one direction. But $\cos r\, dz + r\sin r\, d\theta$ twists in every direction (ie value of θ).

Newer perspective:

[Gromov 1985]: shows that the compactifications (S^3, ξ_0) and (S^3, ξ_{OT}) differ: the 1st one "bounds" a symplectic manifold, but not the 2nd one.

Proof: (S^3, ξ_0) bounds a Darboux (D^4, ω_0). Let's prove that (S^3, ξ_{OT}) does not bound. Suppose it does: it bounds $(W, \omega = d\lambda)$ (take ω exact to be nice). The disk  has a characteristic foliation, $\{z=0, r \leq \pi\}$ looking like . [Bishop]

Start filling by holomorphic disks in 1-param family. Gromov proves that the family persists, until compactness kicks off:

$\to$ Could have bubble (green), but does not exist by Stokes because $(W, d\lambda)$ is exact.

$\to$ ∂ of disks can ~~not~~ be tangent to leaves of characteristic foliation of OT disk by compactness, but this can't happen by the maximum principle.

So one of these 2 must happen, but none can $\Rightarrow$ Contradiction. $\square$

Picture for that local family: 

(1st one is constant)

There is an explicit local model in $\mathbb{C}^2$, for that family of disks. The ∂ of the disks in that family are in D^2_{OT}; by maximum principle, it can not be tangent to ∂D^2_{OT}.

These disks are all disjoint.

No! 

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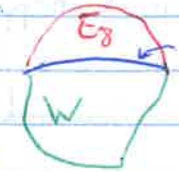
Rem: not every $(2n+1)$-manifold has a filling, like $\mathbb{CP}^2$ in even dim case. In odd dim case: $SU_3/SO(3)$, $\mathbb{CP}^2 \times_{conj} S^1$.
But: $\Omega^{U}_{2n+1} = 0$, i.e. every (Y^{2n+1}, ξ) contact is the boundary of a manifold W^{2n+2} with W almost complex.
↳ unitary bordism ring ↳ or even almost contact
⚠ so far, compatibility between ξ and J is not clear.

From symplectic viewpoint:
We want to distinguish W_1 and W_2 via their contact boundaries.
ex. $V^6 := \{ z_0^7 + z_1^2 + z_2^2 + z_3^2 = 0 \} \subseteq \mathbb{C}^4$; consider $\mathbb{C}^4 \setminus V$, then modify its topology a bit (handle attachment); call the new one $\widetilde{\mathbb{C}^4 \setminus V}$.
[McLean 2008]: $\widetilde{\mathbb{C}^4 \setminus V} \overset{\text{diffeo}}{\cong} \mathbb{R}^8$, is it symplectomorphic to $(\mathbb{R}^8, \omega_0)$?
By looking at the boundaries (with SH^+) (ie contact structure at ∞), you can distinguish them. By connect-summing copies of $\widetilde{\mathbb{C}^4 \setminus V}$, get ∞ly many of those, all not symplectomorphic.
Actually: we distinguish the exact symplectomorphism type.

Q1: what about contact structures with no filling? (Day 3/4)
A1: yes, they exist: Seiberg-Witten invariants are a good obstruction for 3-manifolds.
ex: $\Sigma(2,3,5) = \{ z^2 + x^3 + y^5 = 0 \} \cap S^5 \subseteq \mathbb{C}^3$ is the Poincaré homology sphere.
In fact, $\partial E_8 = -\Sigma(2,3,5)$ ($E_8 = E_8$-plumbing), $\boxed{\text{Smoothly}}$.
Seiberg-Witten: ∂ has positive scalar curvature $\Rightarrow b_2^+(\text{filling}) = 0$.

X^4 

$\Sigma_{2,3,5}$ By smooth topology (Donaldson's diagonalization), X^4 can not exist.

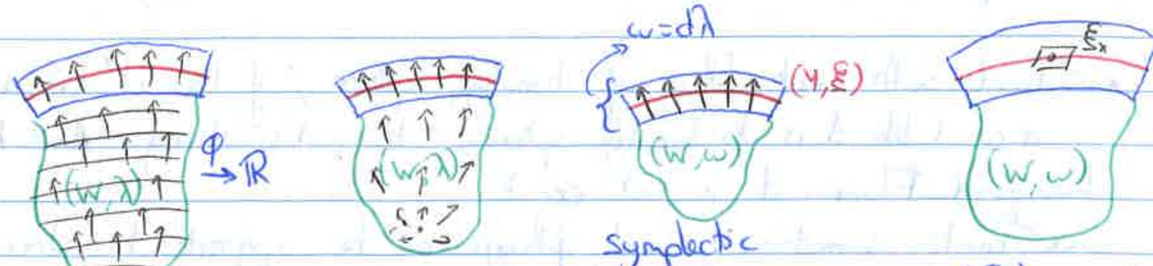
Strange phenomenon: can always find caps:  (Y, ξ). Also, there is a maximum principle, but no minimum principle.

Now, we'll see the different types of fillings.

(Weinstein)

Stein $\subseteq$ Exact $\subseteq$ Strong $\subseteq$ Weak (3D)

(black arrows = dual to λ)



can find $\varphi: W \to \mathbb{R}$ st V is ∇-like. — $\omega = d\lambda$ everywhere — + exact near ∂ — $\omega_x(\xi_x) > 0$, $\forall x$, and orientations should agree.

Stein, exact & strong induce a contact structure on the boundary; for weak, we are comparing ω and ξ.

Rem: for Stein, exact, strong: we can concatenate fillings; this is not the case for weak fillings

⚠ distinction between concave and convex ends in subtle, you can not just switch the sign of the forms.

Q2: are there contact manifolds with infinitely many Stein fillings?
Day 2: yes; we will have (Y^3, ξ) admitting a family X_n^4 of Stein fillings, with $H_*(X_n) = \mathbb{Z} \oplus \mathbb{Z}_n$ ($\Rightarrow$ different).
Sketch: try to factor elements of $\Gamma_{g,n}^k$ (mapping class group of genus g surface) in 2 positive different ways. This uses (day 1) Lefschetz fibrations and open books. This is very algebraic; the only point using Stein-ness is the word "positive" about.

Rem: the inclusions above are all strict! ~~(in 3D)~~
Examples in 3D:
Strong $\not\subseteq$ Weak
* (T^3, ξ_n) with $\alpha_n = \cos(nx)dy + \sin(nx)dz$.

Theorem: (1) ξ_n are all weakly fillable (by $T^2 \times D^2 = T^*T^2$).
(2) Only $\xi_1 = \partial(T^*T^2)$ is strongly fillable.

Proof: (1) is easy. For (2): take Lagrangian $T^2_{\text{Clifford}} \subseteq (\mathbb{R}^4, \omega_0)$; a neighborhood of this is $\simeq T^*T^2$. Consider the complement, and take an n-fold cover; it has n ends (because the complement had 1 end).

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So it looks like $\mathbb{R}^4$ at infinity, hence by Gromov there is only one possibility: $n=1$. So we can not ~~weak~~ strongly fill it if $n \neq 1$.

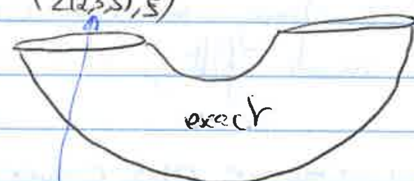
Exact & strong

* Start with weak filling of homology sphere; if the H^* of boundary is nice (like it is for homology spheres), then it's strong. And by Heegaard-Floer: it is not exact.
  ⇝ Trick: sometimes, weak fillings can be upgraded to strong fillings, if the homology of the boundary is nice enough.

Stein & exact

* [McDuff]: constructs exact filling with 2 convex boundaries
  ⇒ there must be a 3-handle
  ⇒ this can not be Stein.

Other example: $(\Sigma(2,3,5), \xi)$ $(\Sigma(2,3,5), \xi_0)$



The exact filling is explicit (convex combination of ξ and ξ_0). Why can we not Stein-fill it? Attach a 1-handle. If $(\Sigma(2,3,5), \xi)$ has a Stein filling, get a contradiction from:

Theorem [Eliashberg] if we have a Stein-filling of $(Y_1^3, \xi_1) \# (Y_2^3, \xi_2)$, then both (Y_i^3, ξ_i) are Stein fillable.

Rem. we have no clue as to what happens in high-dimensions.

ex: $\mathbb{CP}^5$ does not have Stein filling by homotopical reasons, but we don't know if it has an exact filling.

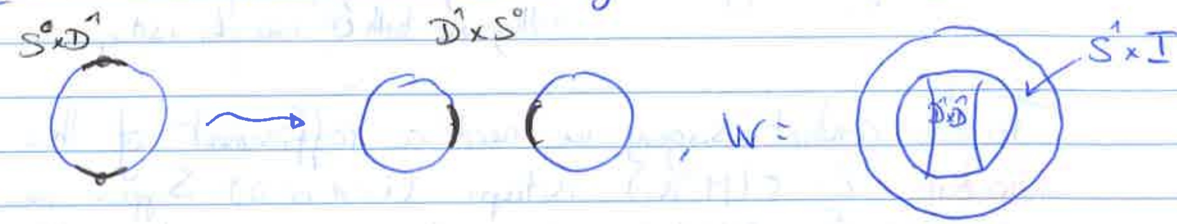
ex. Stein fillings of exotic structures on S^5 are related to hard questions in other areas (s.t. geometric group theory).

François - Simon : Weinstein handles and contact surgery

Smooth surgery:

M^n manifold ; consider $S^k \times D^{n-k} \subseteq M$; think of it as $S^k \subseteq M$ together with a trivialization of its normal bundle.
Observe that $\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$.
$\rightsquigarrow$ get $M' = (M \setminus \mathrm{int}(S^k \times D^{n-k})) \cup (D^{k+1} \times S^{n-k-1})$
Fact: M and M' are cobordant, by $W = M \times I \cup D^{k+1} \times D^{n-k}$.

ex:



Contact surgery:

$(M^{2n+1}, \xi = \ker\alpha)$, let $S^k \subseteq M$ isotropic, with trivialization of the "conformal symplectic normal bundle". We'd like to have M' and W as above, but in a contact / symplectic way.

$\rightarrow$ Isotropic submanifolds: $L \subseteq (M, \xi)$ isotropic if $TL \subseteq \xi$.
$\xi = \ker\alpha \Rightarrow d\alpha|_\xi$ is a symplectic form on ξ
$\xi = \ker(f\alpha) \Rightarrow d(f\alpha)|_\xi = f\, d\alpha|_\xi$, so get a conformal symplectic structure.
If $\eta \subseteq \xi$, then $\eta^\perp = \{x \in \xi \mid d\alpha(x,v) = 0 \ \forall v \in \eta\} \subseteq \xi$ as well.
In our case: L isotropic $\Rightarrow TL \subseteq TL^\perp$.
The conformal symplectic normal bundle: $CSN(L) = TL^\perp / TL$; it has a conformal symplectic structure induced by ξ.

Theorem: let $L_i \subseteq (M_i, \xi_i)$ isotropic ($i = 1$ or 2). Suppose we have

$$\begin{array}{ccc} CSN(L_1) & \xrightarrow{\Phi} & CSN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\varphi} & L_2 \end{array}$$

where Φ is an isomorphism of conformal symplectic structures, and φ is a diffeo.

Then: φ extends to a contactomorphism $\mathcal{O}p(L_1) \to \mathcal{O}p(L_2)$.

Proof: $L \subseteq (M, \xi)$ isotropic: $TL \subseteq TL^\perp \subseteq \xi|_L \subseteq TM|_L$. We have

$$NL \cong TM|_L/\xi|_L \oplus \xi|_L/TL^\perp \oplus TL^\perp/TL$$
$$\overset{\shortparallel}{TM|_L/TL} \cong \langle R_\alpha \rangle \oplus T^*L \oplus CSN(L)$$

The identification $\xi|_L/TL^\perp \cong T^*L$ comes from $\xi|_L \to T^*L : Y \mapsto d\alpha(Y, -)$;

by non-degeneracy of $d\alpha$, it is surjective, and the kernel is by definition $TL^{\perp}$.

Get $\Psi : NL_1 \xrightarrow{\sim} NL_2$: $\begin{cases} R_{\alpha_1} \mapsto R_{\alpha_2} \\ (\phi^*)^{-1} : T^*L_1 \to T^*L_2 \\ \Phi : CSN(L_1) \to CSN(L_2) \end{cases}$

$\Rightarrow$ By tubular neighborhood theorem, get $\tilde{\phi} : \mathcal{O}p(L_1) \to \mathcal{O}p(L_2)$, such that $D\tilde{\phi}|_{L_1} = D\phi \oplus \Psi$, and $\phi^*\alpha_2 = \alpha_1$ on L_1. Make the contact forms agree everywhere by Gray's theorem.

they are both 0, since L_i isotropic. $\square$

To do contact surgery, we need a raffinement of this:

Theorem: $L_i \subseteq (M, \alpha_i)$ isotropic ($i = 1$ or 2). Suppose we have

$$\begin{array}{ccc} SN(L_1) & \xrightarrow{\Phi} & SN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

where Φ is an isomorphism of symplectic bundles, and ϕ is a diffeo.

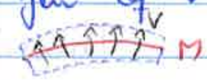
Then, ϕ extends to $\tilde{\phi} : \mathcal{O}p(L_1) \to \mathcal{O}p(L_2)$ st $\tilde{\phi}^*\alpha_2 = \alpha_1$.

Rem: so, here, we fix the forms.

$\rightarrow$ Liouville vector fields: vector field V on (W, ω) such that $\mathcal{L}_V\omega = \omega$ ($\Leftrightarrow d\lambda = \omega$, where $\lambda = \omega(V, -)$).

ex: for $(M, \xi = \ker(\alpha))$, we have the symplectization $(\mathbb{R}_t \times M, d(e^t\alpha))$; then $\partial/\partial t$ is Liouville.

ex: for $M^{2n-1} \subseteq (W, \omega)$ and $V \pitchfork M$ Liouville, we say M is "contact type"; indeed, $\alpha = \lambda|_M$ is a contact form on M, and the flow of V gives an identification $\mathcal{O}p(M) \overset{\text{symp}}{\cong} ((-\varepsilon, \varepsilon) \times M, d(e^t\alpha))$.



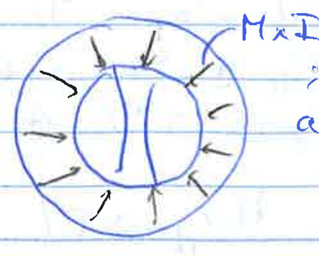
Theorem: $L_i \overset{\text{isotropic}}{\subseteq} M_i^{2n-1} \subseteq (W_i^{2n}, \omega_i)$ ($i = 1$ or 2); $V_i \pitchfork M_i$ Liouville.

If we have

$$\begin{array}{ccc} SN(L_1) & \longrightarrow & SN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

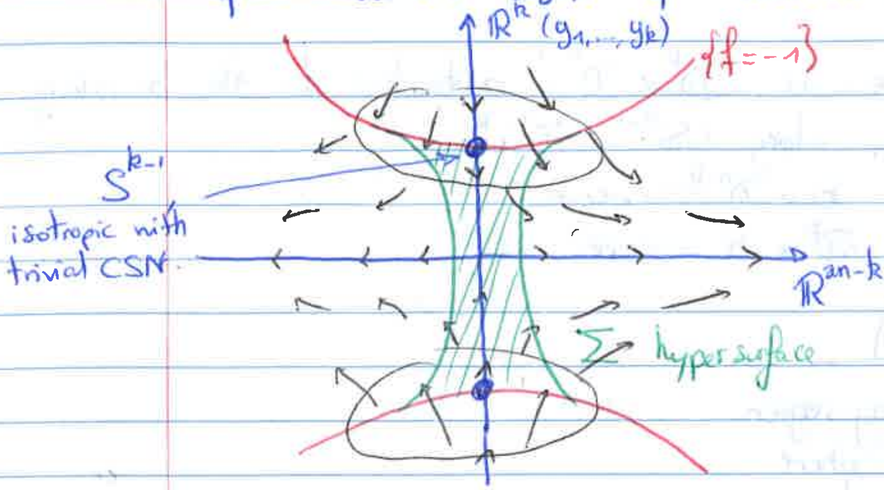
then ϕ extends to a symplectomorphism $\mathcal{O}p_{W_1}(L_1) \to \mathcal{O}p_{W_2}(L_2)$, such that it identifies M_1 with M_2 and V_1 with V_2.

So for contact surgery:

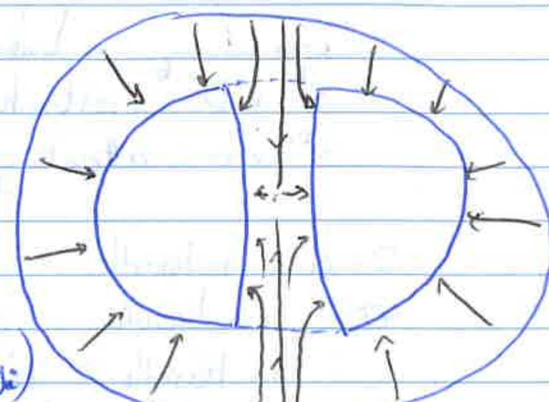


; now we need to build a symplectic handle ("Weinstein" handle).

→ Symplectic (Weinstein) handles: in $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$, consider $f = \sum_{i=1}^{k} (x_i^2 - \frac{1}{2} y_i^2) + \frac{1}{4} \sum_{i=k+1}^{n} (x_i^2 + y_i^2)$, then $V = \nabla f$ is Liouville.



The black circles represent the attaching region of the handle; Σ is a hypersurface such that $\Sigma \pitchfork V$, and the region inside it is the handle.



After gluing that handle, we get:

Rem: $V = \sum_{i=1}^{k} (2x_i \partial x_i - y_i \partial y_i) + \frac{1}{2} \sum_{i=k+1}^{n} (x_i \partial x_i + y_i \partial y_i)$

Just so we do it at least once, let's check it's Liouville:

$\omega(V, -) = \sum_{i=1}^{k} (2x_i dy_i + y_i dx_i) + \frac{1}{2} \sum_{i=k+1}^{n} (x_i dy_i - y_i dx_i)$

$\Rightarrow d(\omega(V,-)) = \sum_{i=1}^{k} (2dx_i \wedge dy_i + dy_i \wedge dx_i) + \frac{1}{2} \sum_{i=k+1}^{n} (dx_i \wedge dy_i - dy_i \wedge dx_i) = \sum_{i=1}^{n} dx_i \wedge dy_i = \omega.$

Idea: can have at most half the directions pointing inwards (the y_i's), because they need to be compensated by the other ones.

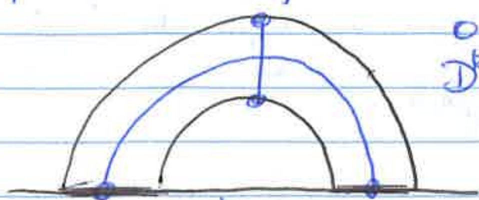
Rem: really what's going on, is that in the half-dimension, we have the holomorphic function z^2; the level sets of the real part and the imaginary part are transverse, by holomorphicity.

Rem: if $k = n$, $SNL = 0$, so we don't have to worry about SNL_1 and SNL_2 being ~~isomorphic~~ symplectomorphic.

Orsola - Kirby calculus for Stein manifolds

Ref: Gompf's paper and book.

An n-dim'l k-handle is $D^k \times D^{n-k}$ attached to the boundary of an n-manifold X, along $S^{k-1} \times D^{n-k}$.

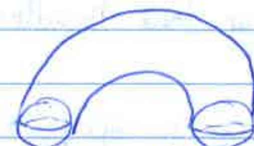


$0 \times D^{n-k}$ = cocore
$D^k \times 0$ = core

$S^{k-1} \times D^{n-k}$ = attaching region
$S^{k-1} \times 0$ = attaching sphere

2-dim 1-handle: 

4D: 0-handle: 

1-handle $D^1 \times D^3$:  $S^0 \times D^3$

notation: 

2-handle: $D^2 \times D^2$: along a circle. If the circle goes through the 1-handle get 
along $S^1 \times D^2$
$f: S^1 \times D^2 \longrightarrow S^1 \times D^2$ is the framing.
$\mu \longmapsto p\mu + q\lambda$; $r := p/q$.

3D: 0-handle: B^3
1-handle: $B^1 \times B^2$, along $S^0 \times B^2$: 

2-handle: $B^2 \times B^1$, along $S^1 \times B^1$:  thick capping.
framings = homeo of annulus $S^1 \times B^1$; there are only 2 of them.

3-handle: B^3

4D: 3-handle: $B^3 \times B^1$, attached along $S^2 \times B^1$
4-handle: B^4

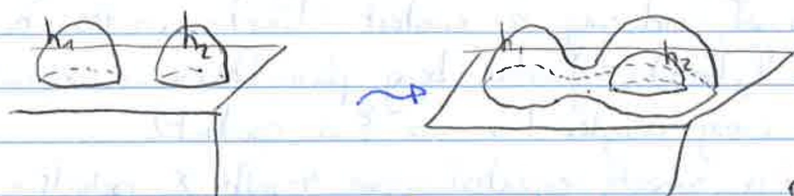
If $\phi(S^{k-1})$ is the attaching sphere, then a framing is $f : \nu\phi(S^{k-1}) \to S^{k-1} \times \mathbb{R}^{n-k}$.

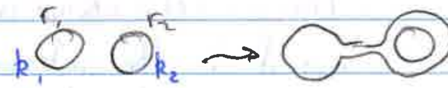
In 4D, we get what's called a Kirby diagram. We'd like to do moves on it, to simplify it.

1) Handle sliding: h_1, h_2 handles of index k. Take the attaching sphere of h_1, and slide it over the other one until it's back on the original manifold.

$\left.\begin{array}{l} k=1 \\ n=2 \end{array}\right\}$:

$\left.\begin{array}{l} k=2 \\ n=3 \end{array}\right\}$.



The attaching circles look like: 

$r_1' = r_1 + r_2 + 2\,lk(k_1, k_2)$

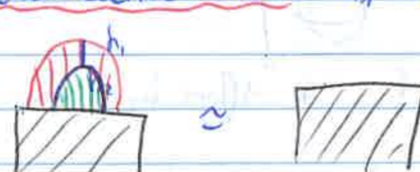
In homology: if these 2 handles give $\alpha_1, \alpha_2 \in H_2(X)$, this move gives $\cancel{(\alpha_1, \alpha_2) \mapsto}$ $\alpha_1 \mapsto \alpha_1 + \alpha_2$.

Rem: the intersection in homology corresponds to the linking number of the attaching spheres.

2) Handle cancellation: h_1 index k, h_2 index $k-1$

The belt sphere of h_1 and the attaching sphere of h_2 intersect once.

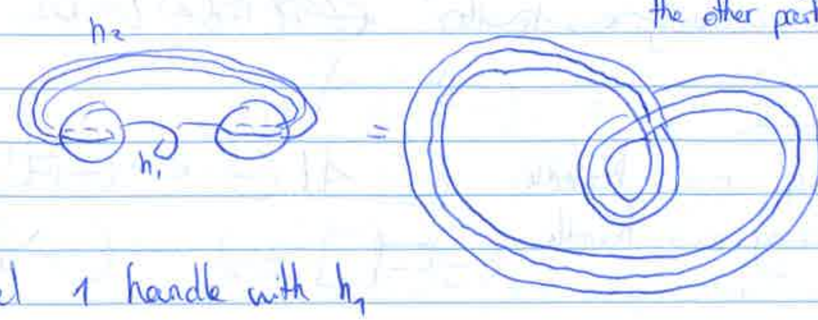
n=2:



n=4: a 1-handle and a 2-handle cancel out if  = nothing.

part of attaching circle of 2-handle, the other part is in the 1-handle from ⊖ to ⊖.

ex.



: slide the 3 2-handles across the 2-handle, and then simplify:

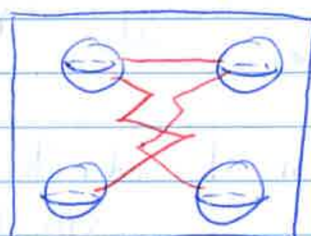
cancel 1 handle with h_1

Rem: we should include the framing for the knots we drawn (which are the attaching circles of the 2-handles). For the Legendrian case, that framing will be canonical.

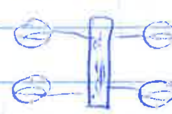
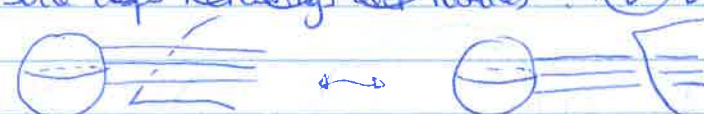
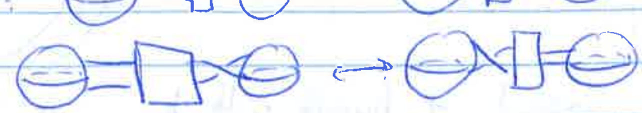
Stein manifolds.

These are the complex manifolds admitting a proper holomorphic embedding into some $\mathbb{C}^N$. Equivalently, it admits an exhausting strictly subharmonic function $f: X \to \mathbb{R}$ ($f^{-1}(c) = \partial f^{-1}((-\infty, c))$).
A Stein domain is a complex compact manifold that admits such an f.
For $\mathbb{C}$-dim $= 2$, these are called Stein surfaces. They have almost complex structures J, inducing a contact structure on the boundary: $\xi = T_p(\partial X) \cap J T_p(\partial X)$ (function being plurisubharmonic means that it is harmonic on every complex line $\Rightarrow \xi$ is contact).

Theorem (Eliashberg) a smooth oriented open 4-mfld X admits a Stein structure $\Leftrightarrow$ it is the interior of some handlebody H such that
(a) every handle has index $k \leq 2$
(b) each 2-handle is attached along a Legendrian knot k_i in ξ on (0-handle $\cup$ 1-handle), and it has framing $tb(k_i) - 1 \in \mathbb{Z}$.

In Kirby diagram:  We need the framing to be $tb(k_i) - 1$.

The canonical framing is $J\tau \in \xi$, it differs by 1 from the product framing.

Theorem [Gompf] K_1 is isotopic to K_2 in "standard form"  if we can go from one to the other by one of 6 moves:
- the Reidemeister moves (3 of them) 
- ~~slide cusps & crossings over handles~~:
- 
- slide cusp over handle: 
- slide crossing over handle: 

Alvin - Lefschetz fibrations and open books

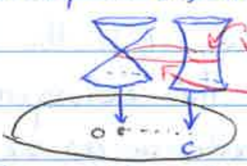
Definition: a Lefschetz fibration on a 4-manifold M is a map $\pi : M \to D^2$ such that

- π has finitely many critical values $t_1, \dots, t_n \in \mathring{D}_n$
- $\exists$ unique critical point $p_i \in \pi^{-1}(t_i)$
- $\exists$ local coordinates near p_i such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Rem: away from these critical values, π is actually a fibration, with fiber F.

Take $\mathcal{U}$ a nbhd of p_i, in which $\pi(x_1+iy_1, x_2+iy_2) = x_1^2+x_2^2-y_1^2-y_2^2+2i(x_1y_1+x_2y_2)$.

For c real, $\pi^{-1}(c) \cap \mathcal{U} = \{(z_1, z_2) \mid x_1^2+x_2^2-y_1^2-y_2^2 = c,\ x_1y_1+x_2y_2 = 0\}$



vanishing cycle : imaginary part of coordinates is 0.

thimble = origin + vanishing cycles

From critical fiber to other ones, we attach a 2-handle. $f = -x_1^2 - x_2^2 + y_1^2 + y_2^2$.

Definition: an open book decomposition of M is a pair (B, π) where

(1) B is an oriented link (the "binding")

(2) $\pi : M \setminus B \to S^1$ is a fibration, such that $\pi^{-1}(\theta)$ is the interior of a compact surface Σ_θ, and $\partial \Sigma_\theta = B$. (the "page")

Definition: an abstract open book is a pair (Σ, ϕ) such that

(1) Σ is an oriented compact surface

(2) $\phi : \Sigma \to \Sigma$ is a diffeo which is the identity in a nbhd of the boundary; it is called the "monodromy"

Rem: can build a 3-mfld : $M_\phi = \underbrace{\Sigma_\phi}_{\text{mapping torus}} \cup_{id} \bigsqcup_{|\partial\Sigma|} S^1 \times D^2$

Example: $S^3 \subseteq \mathbb{C}^2$: $\mathcal{U} = \{z_2 = 0\}$, $\pi_{\mathcal{U}} : S^3 \setminus \mathcal{U} \to S^1 : (z_1, z_2) \mapsto \frac{z_1}{|z_1|}$

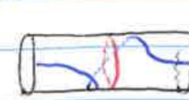
$\leadsto$ ~~bound~~ binding $= S^1$, pages = shaded region.

Example: $\mathcal{U}$ = Hopf link = $\{(z_1, z_2) \mid z_1 z_2 = 0\}$; $\pi : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$

Theorem: every closed oriented 3-mfld has an OBD.

Theorem [Giroux] if M is closed oriented 3-mfld, then $\exists$ bijection

$$\{\text{oriented contact structures}\}/\text{isotopy} \longleftrightarrow \{\text{OBD}\}/\text{stabilization}$$



Stabilization: attach 1-handle, and compose monodromy with right-handed Dehn twist.

Now, consider a Lefschetz fibration $\pi : M \to D^2$; it has
* a vertical boundary : corresponds to F over ∂D
* a horizontal boundary : union of boundary of fibers

Every $x \in \partial M$ lies in one of these 2. This gives an open book: The fibers of the vertical boundary are the pages, and the horizontal boundary is a nbhd of the binding.

Rem: for every critical point, attach a 2-handle along a vanishing cycle. In ∂: ~~vanishing~~ Dehn surgery along vanishing cycle.

Clearer: start with regular point; a nbhd consists of regular fibers. For each crit value of π, attach a 2-handle, corresponding to the thimble for that critical value. On the boundary, the effect of this is a Dehn twist along the corresponding vanishing cycle: this Dehn twist is the new monodromy as we go along $S^1 = \partial D^2$.

{Stein fillings}/deformation $\longleftrightarrow$ {Lefschetz fibrations}/stabilization.

Fix contact manifold; look at OBD and factor the monodromy into right-handed Dehn twists.

Discussion session

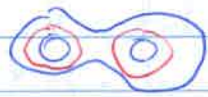
Game: Σ surface with boundary, $\mathcal{C} = \{\text{curves in } \Sigma\}$
Construction: start with $\Sigma \times D^2$, then attach h_2^4 along the curves in $\mathcal{C}$.

1) $\Sigma =$ (disk), $\mathcal{C} = \phi \rightsquigarrow$ get D^4
2) (annulus), $\phi \rightsquigarrow$ get $T^*S^1 \times D^2 = S^1 \times D^3$
3) (annulus with curve) $\rightsquigarrow \chi$: 1 0-h, 1 1-h, 1 2-h $\Rightarrow \chi = 1$

(annulus) = (disk) $\cdot$ (disk) ; the red curve is (disk)—(disk) $\Rightarrow$ get $\mathbb{C}^2$

4) (annulus with curves) $\rightsquigarrow$ (disk)═(disk), get T^*S^2
If it goes around 3 times: plumbing $T^*S^2 \# T^*S^2$

5) If 2 1-handles: (pair of pants) or (cylinder), but they are the same when we cross with D^2



: get 2 connect sum of (annulus), so $\mathbb{C}^2$ again.

: $\chi = 0$ the red line (2-handle) cancels the left 1-handle.

$=$ (annulus) $= S^1 \times D^3$

The other way around: (Σ, λ) Liouville ; $(\Sigma \times D^2, \lambda + \lambda_0)$ is Stein.

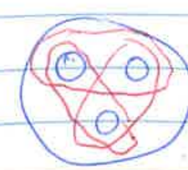
$$\begin{array}{ccc} \Sigma \times D^2 & \xrightarrow{\text{attach as a critical point}} & X^4 \\ \downarrow & & \downarrow \\ D^2 & & D^2 \end{array}$$

; vanishing cycles correspond to the handles of the critical points

The monodromy around one of the critical values is a Dehn twist.

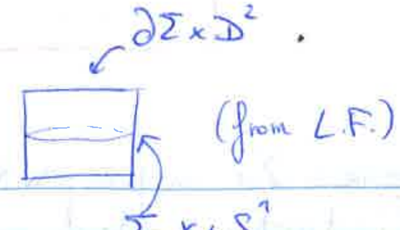
Theorem: any Stein X^4 admits such a Lefschetz fibration.

Ex: inverse game
1) T^*S^2: (annulus) $\quad T^*S^2 = \{z^2 + w_1^2 + w_2^2 = 1\} \xrightarrow{z}$ (x x)



2) $T^*T^2 = \underset{(x,y)}{(\mathbb{C}^*)^2} \quad (x,y) \mapsto x + \frac{1}{x} + y + \frac{1}{y}$, or $x + y + \frac{1}{xy}$.
$\downarrow$
3-punctured torus

3) $T^*\mathbb{RP}^2$ = complement of projective conic in $\mathbb{CP}^2$
$\rightarrow$ 4-punct. sphere.



Now, it's open books time!
Take (Σ, λ) Liouville, and ϕ a symplectomorphism ($\phi|_{Op(\partial\Sigma)} = id$).
Let's build a contact 3-mfld out of it. Smoothly, get
$\underbrace{\Sigma \times_\phi S^1}_{\lambda + d\theta} \sqcup \underbrace{\partial\Sigma \times D^2}_{ds + r^2 d\theta}$ (s coord. on $\partial\Sigma \simeq S^1$, or a bunch of it)
if no d. $\to$ for $(\cdot) \times_\phi (\cdot)$: it's ok, can make it work.

Theorem [Giroux] any (Y^3, ξ) has an adapted OB: $(Y^3, \xi) \simeq OB(\Sigma, \phi)$.
(the contact planes will be almost tangent to Σ)

ex: $(Y^3, \xi) = OB(T^*S^1, \tau^2_{S^1})$; we know it is the boundary of the LF with total space T^*S^2 $\leadsto$ get $\mathbb{RP}^3$.
ex: $(Y^3, \xi) = OB(T^*S^1, \tau^{-1}_{S^1}) = (S^3, \xi)$ (actually, $\xi = \xi_{OT}$)
ex: Σ closed, $\omega \in H^2(\Sigma; \mathbb{Z})$; take the bundle with that ~~class~~ Euler class.

cf in , tb of the red circle is 0 $\to$ OT disk

$\mathcal{O}(k) \xrightarrow{\partial} Y_k$

$\downarrow \qquad\qquad \downarrow S_1$

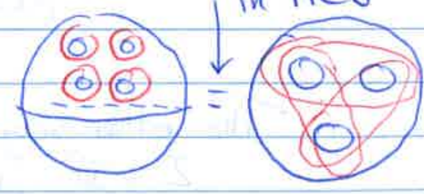
lens space

$Y^3_1 = S^3$, $Y^3_k = L(k,1)$ or $L(-k,1)$

$\downarrow \qquad\qquad \downarrow$

$S^2 \qquad\qquad S^2$

lantern relations in MCG



Page = k-punctured sphere, monodromy is positive Dehn twists around each of the punctures
And there are different 4-manifolds, because they have different χ.

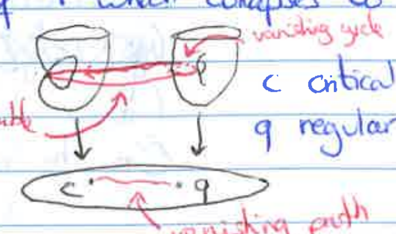
Bahar - Mapping class group factorizations as Lefschetz fibration fillings

Goal: to study how the topology of the total space of an exact Lefschetz fibration (LF) is described by using a distinguished basis of vanishing paths.
Main ingredients: LF and OBD's monodromy of the boundary of the total space.

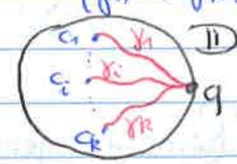
Lefschetz fibration: $f: W^4 \to \mathbb{D}^2$ such that
(a) $\mathrm{crit}(f)$ are isolated non-degenerate in $\mathrm{int}(W)$.
(b) local behaviour around critical points: $(z_1, z_2) \mapsto z_1^2 + z_2^2$
(c) for $F = f^{-1}(q)$ for q regular value, $\omega = \omega_F + K \cdot f^*(d\sigma)$, for K large
canonical sympl. form on $\mathbb{D}^2$
Recall: $\partial_v W = f^{-1}(\partial \mathbb{D}^2)$ and $\partial_h W = \coprod_{z \in \mathbb{D}^2} \partial F_z$

Motivation: they provide some manageable approach to studying symplectic 4-mflds.
Fix a point $q_* \in \mathbb{D}^2 \setminus \mathrm{crit}(f)$, and let $F = f^{-1}(q_*)$ be the corresponding fiber. Then, we can consider $\Psi: \pi_1(\mathbb{D} \setminus \mathrm{crit}(f), q_*) \to \pi_0(\mathrm{Diff}^+(F))$.
$\gamma \mapsto \{$isotopy classes of diffeos of F fixing ∂F given by parallel transport along $\gamma\}$.
It is the monodromy of the LF f.

Definition: a vanishing cycle is a homology cycle of F which collapses to the unique singular point of the critical fiber under the parallel transport. The corresponding thimble is the image of the entire parallel transport.
The vanishing path is the image of the thimble under f.
Rem: really, choose a path first, and then get a thimble.

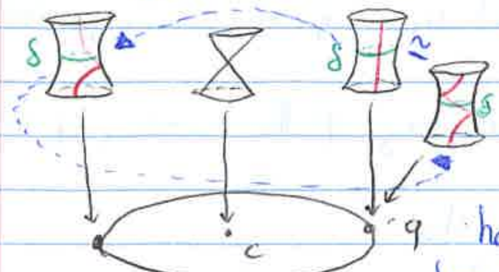


Definition: a distinguished basis of vanishing paths $\{\gamma_1, \dots, \gamma_k\}$ is an ordered set of vanishing paths, for each critical value c_i of f, starting at q ($= q_*$ above) and ending at the critical value c_i.



Rem: nowadays, we should think of the vanishing cycle as a Lagrangian, instead of a homology class. That's what symplectic geometry buys us.

Recall: given a symplectic LF: $W \to \mathbb{D}^2$, $f^{-1}(\partial W)$ is a F-bundle over S^1 $\simeq F \times [0,1] / (1,x) \sim (0, \Psi(x))$ ($=$ mapping torus of OBD of ∂W), Ψ is the geometric monodromy. Moreover, the monodromy (around one critical point) of a LF is a (right-handed) Dehn twist.



$D(\delta) :=$ Dehn twist along δ.

For critical values $c_1, \dots, c_n$ along with a distinguished basis of vanishing paths $\gamma_1, \dots, \gamma_k$, have $\psi = D(\delta_1) \dots D(\delta_k) \in \mathrm{Map}\, F = \pi_0(\mathrm{Diff}^+(F, \partial F))$, where $\delta_i :=$ vanishing cycle over γ_i.

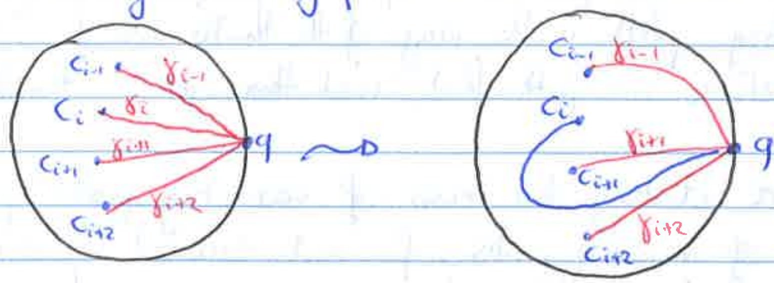
To every LF, we can associate a monodromy factorization in $\mathrm{Map}(F)$

<u>Fact</u>: the monodromy factorization depends on
(1) the topology of the fiber F.
(2) the choice of the vanishing paths.

Let $(\gamma_1', \dots, \gamma_k')$ be another distinguished basis of vanishing paths of f. Then, it induces a different monodromy factorization.
<u>Q</u>: How are they related?
<u>A</u>: Hurwitz moves: every 2 dist. bases of van. paths are related via elementary Hurwitz moves: on vanishing cycles, it looks like:

$(\delta_1, \dots, \delta_{i-1}, \delta_i, \delta_{i+1}, \dots, \delta_n) \mapsto (\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, D(\delta_{i+1})(\delta_i), \delta_{i+2}, \dots, \delta_n)$

On the level of vanishing paths:

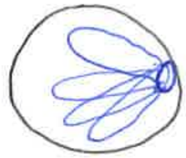


<u>Rem.</u> in general, $f^{-1} \tau_b f = \tau_{f(b)}$. $(*)$

For monodromies: $D(\delta_1) \dots D(\delta_k)$

$$D(\delta_1) \dots D(\delta_{i-1}) \overbrace{D(\delta_{i+1}) D^{-1}(\delta_{i+1})}^{id} D(\delta_i) D(\delta_{i+1}) \dots D(\delta_k)$$

we see they are equal. $\quad D^{-1}(\delta_{i+1}) D(\delta_i) D(\delta_{i+1}) = D(D(\delta_{i+1})(\delta_i))$ by $(*)$



If we think of the thimbles as the cores of the handles we attach, this move corresponds to a handle slide. This corresponds as summing the generators corresponding to these 2 handles on homology, which is what a Dehn twist does.

Note: the classification of LF over the disk amounts to that of monodromy factorization in Map(F) up to Hurwitz equivalence.

Theorem: let F be a regular fiber of a LF $f: W^4 \to \mathbb{D}$ with genus g and n boundary components. If $2-2g-n<0$, then

$$\left\{\begin{array}{l}\text{factorization of boundary Dehn}\\ \text{twists as a product of right-handed}\\ \text{Dehn twists}\end{array}\right\} \Big/ \text{Hurwitz moves} \xleftrightarrow{1-1} \left\{\begin{array}{l}\text{genus } g \text{ with } n\ \partial \text{ components}\\ \text{LF over } \mathbb{D}\end{array}\right\} \Big/ \text{isotopy}$$

ex: F =  $(g=1, n=1)$. We know $Map(T^2) = SL(2;\mathbb{Z})$. It turns out that $Map(F) = \widetilde{SL}(2;\mathbb{Z}) \simeq B_3 = \langle \tau_a, \tau_b \mid \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \rangle$.
$\tau_c = (\tau_a\tau_b)^6$ central element, generates the kernel of $Map(F) \to SL(2;\mathbb{Z})$.

Laura: Factorization of ∂ Dehn twists (g,n) $\longleftrightarrow$ Genus g Lefschetz fillings of corresponding $OBD_{g,n}$

$\updownarrow$ [Giroux]

$(Y_{g,n}, \xi_{g,n})$

where $\begin{array}{c} S^1 \hookrightarrow Y_{g,n} \\ \downarrow \\ \Sigma_g \end{array}$ is the Euler class $=n$ circle bundle.

Theorem [Auroux] the monodromy factorizations

$\Psi: (\tau_a^{-3}\tau_b\tau_a^{3})(\tau_b)(\tau_a^{-3}\tau_b\tau_a^{-3})(\tau_a)$ and

$\Psi: (\tau_a^{-2}\tau_b\tau_a^{2})(\tau_b)(\tau_b)(\tau_a^{2}\tau_b\tau_a^{-2})$ of Ψ in $\widetilde{SL}(2,\mathbb{Z}) = Map(F)$

define inequivalent genus 1 LF's $f_i: W_i \to \mathbb{D}$. The corresponding Stein filling W_1, W_2 of the OB with monodromy Ψ are distinguished by their first homology: $H_1(W_1)=0$, $H_1(W_2)=\mathbb{Z}_2$.

Rem: $H_1(W_i) = H_1(F_i) \big/ \text{span of vanishing cycles}$

ex: $M^3 = OB(F, \Psi)$

$\hookrightarrow$ 

has infinitely many Stein fillings distinguished by their 1^{st} homology classes. [Baykur - Stipsicz]

Rem: # Stein fillings of Lens spaces : $|(L(p,q), \xi_{p,q})| < \infty$.
[Eliashberg - Lisca - McDuff]

Roberta - Pseudo-holomorphic sh*t

Why? **Proposition**: given (M,ω), $\exists$ J compatible almost-$\mathbb{C}$-structure. Moreover, the space of those is contractible.

Uses of J-hol. curves:

- Pick a J and use it to construct some curves, and directly use these curves (ex: proof of OT $\Rightarrow$ non-fillable)
- Use J to build invariants of (M,J); check independance on J, so we get invariants of (M,ω).

Definition: given (M,J) almost complex, a J-hol curve is a map $u:(\Sigma,j)\to(M,J)$, where (Σ,j) is a Riemann surface, such that $J\circ du = du\circ j$

Good properties:

* Unique continuation: if $u_1(z)=u_2(z)$ and all derivatives agree, then $u_1=u_2$
* $\mathrm{Crit}(u)$ is finite
* Either (a) u factors through a branched holomorphic n-cover of Σ_g, or
  (b) the set of points where u is injective is open and dense in Σ_g.

When it is (b), we say that u is simple. So, a J-hol. curve always factors through a branched cover followed by a simple curve.

Moduli space of J-hol. curves:

Can we deform u to get something J-hol? In how many directions? Convergence of sequences?

Definition $\mathcal{M}^*(M,J) := \{\text{all simple J-hol. curves}\}$. We can add restriction. $\mathcal{M}(A,J) = \{u:\Sigma\to M,\ J\text{-hol} \mid u_*[\Sigma]=A\}$, for $A\in H_2(M)$.

Transversality: is $\mathcal{M}(A,J)$ a manifold? We can see it as the 0-level set of the functional $F(u) = J\circ du - du\circ j$, defined on $C^\infty(\Sigma,M)$.
$\hookrightarrow$ is 0 a regular value of F?
Intuition from finite-dim geometry: up to deforming f, 0 can be made a regular value. In our case, for generic J, 0 is a regular value, which means $\mathcal{M}^*(A,J)$ is a manifold.
$\triangle$ $\mathcal{M}^*(A,J)$ = simple curves in $\mathcal{M}(A,J)$.

(Virtual) dimension of $\mathcal{M}^*(A,J)$: when J is regular,

$\dim \mathcal{M}^*(A,J) = \dim(\ker(D_u F))$

$= \operatorname{ind}(D_u F)$ (Fredholm index), since we assume $D_u F$ is surjective

$= n(2-2g) + 2c_1(A)$ $(\dim M = 2n)$

If not transverse, we still call this number the virtual dimension.

Automatic transversality: no need to modify J.

* $\dim = 4$
* not all of $\mathcal{M}^*(A,J)$; only in a neighborhood of $u \in \mathcal{M}^*(A,J)$, or some subset $S \in \mathcal{M}^*(A,J)$.

Theorem [McDuff] if J is integrable in a nbhd of the image of u in M and $c_1 > 2(g-1)$, then u is a regular point.
~~Here, $c_N = c_1(N_u(\text{int}\,u))$~~. And u must be ~~simple~~ embedding.

Theorem [Wendl]: $u : (\Sigma_g, \text{some punctures}) \to M$ has ends on non-degenerate Reeb orbits (of the boundary), then u is regular if $2g - 2 + h_+ <$ Fredholm index.
h_+ → # positive punctures

Reparametrization:
$\mathcal{M}(A,J) := \{\text{all couples } (\Sigma_g, u : \Sigma_g \to M)\} / G$, $G = \{\text{autom of } \Sigma_g\}$
So, $u_1 \sim u_2$ if $u_1 = u_2 \circ f$ for $f : \Sigma_g \to \Sigma_g$ biholomorphic.

Evaluation map: we can't define it on $\tilde{\mathcal{M}}$ since we reparametrize.
Instead, define it on $(\mathcal{M}(A,J) \times \Sigma_g)/G$, as
$ev(u,z) := u(z)$, and the action of G is $(u,z) \sim (u \circ f^{-1}, f(z))$
So now, can study subspaces
$\mathcal{M}(A,J,z_1,\dots,z_k) = \{\text{non-param. curves going through } z_1,\dots,z_k \in M\}$

ex: Gromov-Witten invariants:
$GW(A,J,z_1,\dots,z_k) = \# \mathcal{M}(A,J,z_1,\dots,z_k)$, when $\dim \mathcal{M}(\dots) = 0$.
Rem: only works if $\tilde{\mathcal{M}}(\dots)$ is also compact.

Compactness [Gromov]

What do sequences in $\tilde{\mathcal{M}}^M(A,J)$ approach?

Sequences can diverge, if the "energy" goes to ∞, where $E(u) = \int_\Sigma u^*\omega$

Rem: in $\tilde{\mathcal{M}}(A,J)$, the energy is fixed, as ω is closed.

If the energy is bounded, we can say that there is a "convergent" subsequence, where the limiting curve is allowed to have bubbles:

$u: \Sigma_g \cup \mathbb{CP}^1 \cup \ldots \cup \mathbb{CP}^1 \to M$

 $\longrightarrow M$

So $\tilde{\mathcal{M}}(A,J)$ is not compact, but:

Theorem: $\tilde{\mathcal{M}}(A,J) \cup \{\text{all } u: \Sigma_g \cup \mathbb{CP}^1 \cup \ldots \cup \mathbb{CP}^1 \to M\}$ is compact.

Rem: there are versions of this with punctures, with point constraints, ...

Positivity of intersections:

Let u_1, u_2 be J-hol. in M^4.

Theorem [Gromov-McDuff] The multiplicity of intersection is a positive number at each intersection point.

Corollary: $C_1 \cdot C_2 = 0 \Leftrightarrow$ they are disjoint.

Emily – McDuff's rational ruled classification

Plan: 1) Motivation and context
2) Ideas of proof
3) An application to fillings

1) Aim: classify compact symplectic 4-manifolds containing a symplectically embedded copy of S^2, C, with nonnegative self-intersection.

Why? Fillings! To classify fillings Z of X

$\updownarrow$

Classify compact manifolds V containing a cap.

Containing C $\Rightarrow$ strong restrictions on what V can be
$\Rightarrow$ " " " " Z can be.
(the cap contains the 2-sphere C)

Terminology: (V,ω) compact smooth symplectic 4-manifold.
C rational curve (i.e. an symplectically embedded S^2)
An exceptional curve is a rational curve with self-intersection -1.
We call (V,C,ω) minimal if $V\setminus C$ contains no exceptional curve.
(V,ω) is rational if it contains a rational curve, and ruled if it's fibrated by rational curves.

Theorem: [McDuff '90] If (V,C,ω) is minimal, and $C\cdot C \geq 0$, then (V,ω) is symplectomorphic to (1) $(\mathbb{CP}^2, \omega_{FS})$
(2) a symplectic S^2-bundle over compact M.
Moreover, C is taken to (1) complex line or quadric
(2) fibre or section of bundle.

Rem: * easy to have negative curves, by blowing up
* only classify minimal pairs. Actually, if $V\setminus C$ contains a (-1)-curve, we can blow it down:
Recall blow up:

$M \longrightarrow M \# \overline{\mathbb{CP}}^2$

Theorem [McDuff] every (V,C,ω) covers a minimal $(\bar{V},\bar{C},\bar{\omega})$, obtained by blowing down a finite collection of exceptionnal curves in $V\setminus C$. Given these, $\bar{\omega}$ is unique up to isotopy.

Upshot: the classification problem boils down to the minimal case.

2) Main tool: the adjunction formula. Let S be a rational embedded holomorphic curve. Then $TV|_S = TS \oplus \nu S$

$$\Rightarrow c_1(TV|_S) = c_1(TS) + c_1(\nu S)$$

$$\Rightarrow c_1(V)\cdot[S] = \chi(S) + [S]\cdot[S] \quad \leftarrow \text{adjunction formula}$$

Mega lemma: $\exists$ tame J such that $[C]$ may be represented by a J-hol cusp curve $S = S_1 \cup \dots \cup S_m$, where $[S_i] = A_i$ is J-indecomposable, and J is regular for all A_i-curves. Moreover, the S_i's are distinct embedded curves with self-intersection -1, 0 or 1, and there is at least one component with $A_i \cdot A_i \geq 0$.

Rem. J-indecomposable $\Rightarrow$ moduli space of curves in class A is compact.

Rem: embedded $\Rightarrow$ can use adjunction formula.

Proposition 1: if F is a rational B-curve in (V,ω) where B is simple and $B\cdot B = 0$ then $\exists \pi: V \to M$ compatible with ω (ie ω is non-degenerate on the fibers), with F a fibre.

Proof: pick J a tame a-$\mathbb{C}$-str. which splits near F. If f is a J-hol. parametrization of F, then $c_1(\partial F) = 0$ (since J splits).

Automatic transversality $\Rightarrow (f, J)$ is regular.

B simple $\Rightarrow \underset{\text{genus}\ \ \#\text{marked points}}{\mathcal{M}_{0,1}}(J,B)$ is compact! (we just keep track of 1 marked point, no condition on it)

The dimension is $\underset{g=0}{2n} + 2c_1^{(c)} + \underset{\text{marked pts}}{2k} - \underset{\dim PSL(2;\mathbb{C})}{6}$

By adjunction, $c_1(C) = \chi(S) + [S]\cdot[S] = 2 + 0 = 2$. So, our dimension is $4 + 4 + 2 - 6 = 4$.

Now, consider $ev: \mathcal{M}_{0,1}(J,B) \to V$, map between 4-dim spaces. Since $B\cdot B = 0$, there is at most one B-curve through each point in V (by positivity of intersection). We already have the curve F. $\Rightarrow \deg(ev) \leq 1$.

In a nbhd of F, there is a family of such J-hol curves, so the degree of ev is ≥ 1. Rem: 2-dim family, without the marked points.

$\Rightarrow$ Degree is exactly 1. So there is exactly 1 J-hol B-curve through every point of V. These form the fibers of a cont. surjection $\pi: V \to M$.

Fibers are holomorphic $\underset{\text{tame}}{\Rightarrow}$ symplectic, so π is compatible with ω. $\square$

(26)

Rem: "B is simple" is very often satisfied: any class B with $B.B=0$ in V minimal is J-simple for almost every J.

Rem: ω compatible with π ⇒ ω determined up to isotopy by $[\omega]\in H^2(V)$

Proposition 2: (V,C,ω) minimal, $C\cdot C\geq 0$. Then, (in notations of the mega lemma)

(1) If $S_i\cdot S_i=1$ for some i, then $V\cong\mathbb{CP}^2$ and $m=1$ or 2.

(2) If $S_i\cdot S_i=-1$ for some i, then $V\cong\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ and there is only one such i, and all other S_j's are homologous with $S_j\cdot S_j=0$.

(3) If $S_i\cdot S_i=0$ $\forall i$, then all S_i's (except maybe 1) are homologous, and V is an S^2-bundle (the base of S^2-bundle)

Comments on proof:

(1) Consider $ev: \mathcal{M}_{0,2}(J,A)\to V\times V$; it has degree 1, so there is a unique J-hol A-curve through each pair of points.

 ⟿ construct $\mathbb{CP}^2$ using these lines through p that fill in the whole manifold.

(2) Adjunction ⇒ contradiction. (3) also. □

Propositions 1 & 2 ⇒ diffeo type of V, in McDuff's theorem.
The symplectomorphism type determined just by $[\omega]\in H^2$ up to isotopy

Corollary: diffeo type determined by $p=C\cdot C\geq 0$ for $p\neq 0$ or 4.

$p=0$ ⇒ S^2-bundle, by proposition 1 (but we don't know over what)

$p=1$ ⇒ $\mathbb{CP}^2$ by Gromov, and C is sent to a complex line.

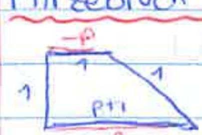
$p\geq 2$: $p=C\cdot C=(A_1+\dots+A_m)^2=\sum A_i^2+\sum_{i<j}2A_i\cdot A_j$. So in case (2) above: $p=-1+\text{even}=\text{odd}$

(3) : $p=0+\text{even}=\text{even}$

So, p odd ⇒ $V\cong\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ (the non-trivial S^2-bundle over S^2)

$p\geq 2$ even ⇒ $S^2\times S^2$ (uses ≥ 2 ✗: there must be a section amongst the S_i's, by the "except maybe 1" before

Hirzebruch surfaces: $\mathbb{F}_p\cong\mathbb{F}_{p+2}$ $\forall p$, but no 2 are symplectomorphic.



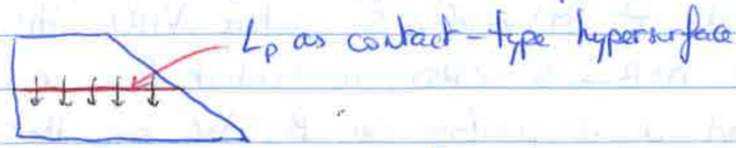
ex: $\mathbb{F}_0=(S^2\times S^2,\omega_{std})$ $\mathbb{F}_1=(\mathbb{CP}^2\#\overline{\mathbb{CP}^2},\omega)$

$\mathbb{F}_2=(S^2\times S^2,\text{other }\omega)$ → self-intersection 4: 

$p=4$: $(S_1+S_2)^2=4$, can get $(\mathbb{CP}^2,\text{quadric})$ also.

3) We can get $\mathbb{F}_p$ by gluing $\mathcal{O}(p)$ to $\mathcal{O}(-p)$ (or rather, the disk bundles).

<u>Claim:</u> $\partial(\mathcal{O}(p)) = L_p = L(p,1)$
$\partial(\mathcal{O}(-p)) = L_p$ with opposite orientation.



We can see that using Heegaard decomposition.

<u>Punchline:</u> suppose we want to classify fillings of L_p.
Stick on a cap $\mathcal{O}(p)$; this will contain a (-1)-curve.
By McDuff, minimal such compact manifolds are determined uniquely up to symplectomorphism if we fix $[\omega]$, and up to diffeo for $p \neq 0$ or 4.

$\Rightarrow L_p$ have minimal symplectic fillings.
For $p \neq 4$, these are unique up to diffeomorphism.
If we fix $[\omega]$, unique up to symplectomorphism.
(and L_4 has exactly 2 non diffeomorphic fillings)

⚠ McDuff classifies pairs (V, C). It is important to know where C goes; for example, it should not go through the L_p, in the example above.

Agustín - Strongly fillable contact manifolds and J-holomorphic foliations

A contact manifold (M,ξ) is planar if it admits an OBD supporting ξ with genus 0 pages. A LF is allowable if all vanishing cycles are homologically non trivial. If $\pi: M\setminus B \to S^1$ is an OBD, denote $\hat{\pi}: M\setminus N(B) \to S^1$, where $N(B)$ = nbhd of B.

Definition: for $\pi: M\setminus B \to S^1$ OBD, a contact form α is Giroux if $d\alpha|_{page} > 0$ and α is positive on B. We say that $\xi = \ker\alpha$ is supported by the OBD.

Rem: it is the case if ξ is almost tangent to the pages.

Reference: Wendl's paper "[title of talk]"

Theorem 1: Suppose (W,ω) is a strong symplectic filling of a planar contact manifold (M,ξ), and $\pi: M\setminus B \to S^1$ is a planer OBD supporting ξ. Then, $\exists$ some enlarged version of W, called (W',ω), obtained by a trivial symplectic cobordism to W, such that W' admits a symplectic LF $\Pi: W' \to \mathbb{D}$ such that $\Pi|_{\partial W'\setminus N(B)} = \hat{\pi}$. Moreover, $\Pi: W' \to \mathbb{D}$ is allowable if W is minimal.

Corollary 1: every strongly fillable planar contact manifold is also Stein fillable.

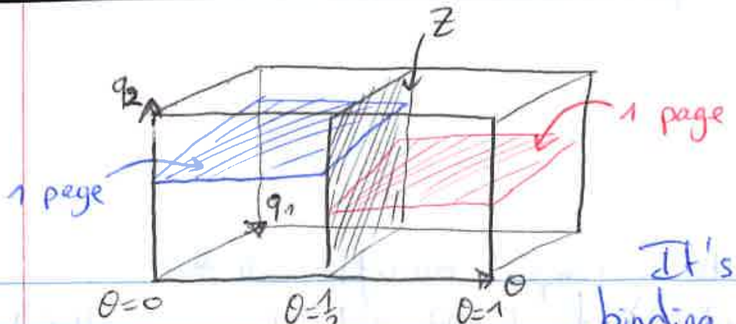
So: strongly but not Stein fillable $\Rightarrow$ not planar.

Theorem 2: Suppose (W,ω) strong symplectic filling of (T^3, ξ_0), where $\xi_0 = \ker(\cos(2\pi\theta)dq_1 + \sin(2\pi\theta)dq_2)$. Then, one can attach to W a trivial symplectic cobordism such that the enlarged filling W' admits a symplectic LF $\Pi: W' \to [0,1]\times S^1$ for which $\Pi|_{\partial W'\setminus N(\hat{Z})} = \hat{\pi}_0$. Moreover, every singular fiber is the union of an annulus with an exceptional sphere; in particular, if (W,ω) is minimal, there are no singular fibers.

$T^3 = S^1\times S^1\times S^1 = T^2\times S^1 \ni (q_1,q_2,\theta)$

$Z = \{\theta \in \{0,1/2\}\} \subset T^3$, and π_0 is

$$\pi_0 : T^3\setminus Z \to \{0,1\}\times S^1 : (q_1,q_2,\theta) \mapsto \begin{cases} (0,q_2) & \text{if } \theta\in(0,1/2) \\ (1,q_2) & \text{if } \theta\in(1/2,1) \end{cases}$$



It's kind of like an OB, but the binding is a T^2, that we should see as a "blow up" of S^1.

Wendl's paper numbering

<u>Theorem 4</u>: all minimal strong fillings of T^3 are symplectically deformation equivalent, and every exact filling of T^3 is symplectomorphic to a star-shaped domain in (T^*T^2, ω_0), ie something of the form $\{(q, t f(q,p) p) \in T^*T^2 \mid t \in [0,1], (q,p) \in S^*T^2, f: S^*T^2 \to (0,\infty)\}$.

<u>Corollary</u>: every minimal strong filling of T^3 (in particular, Stein fillings) is diffeomorphic to $T^2 \times D^2$.

<u>Theorem 5</u>: the group $\mathrm{Symp}_c(T^*T^2, \omega_0)$ of compactly supported symplectomorphisms of (T^*T^2, ω_0) is contractible.

Now, an important compactness theorem, for all these results:

<u>Definitions</u>: * a <u>finite energy foliation</u> $\mathcal{F}$ on $(M_0, \lambda$ contact form, J a $\mathbb{C}$-str$)$ is a foliation of $\mathbb{R} \times M_0$, such that

1) $\forall u \in \mathcal{F}$ leaf, any $\mathbb{R}$-translation of u is also a leaf.
2) every $u \in \mathcal{F}$ is the image ~~is the image~~ of a finite energy J-hol. curve satisfying some uniform energy bound.

* A leaf $u \in \mathcal{F}$ is <u>interior</u> if it is not a trivial cylinder and all its ends belong to Morse-Bott submanifolds in M_0.
* $\mathcal{F}$ is <u>positive</u> if every leaf $\neq$ trivial cylinder has only positive ends.
* $u \in \mathcal{F}$ is <u>stable</u> if it has genus 0, all punctures are odd, and its index is 2. (1-param for $\mathbb{R}$ translation, 1 for family on the base)
* $u \in \mathcal{F}$ is asymptotically simple if all of its asymptotics are simply covered, and belong to pairwise disjoint Morse-Bott submanifolds.

<u>Holomorphic curves & compactness</u>.

(M^3, ξ) with Morse-Bott λ, J_+ ~~a~~ λ-compatible

$M_0^\varepsilon \subseteq M$ compact submanifold with Morse-Bott boundary

$\mathcal{F}_+$ positive finite energy foliation on (M_0, λ, J_+) containing an interior stable leaf that is asymptotically simple

Also, let (W^∞, ω) be a compact manifold such that
$W^\infty = W \cup_{\partial W} ([R,\infty) \times M)$ for $R \in \mathbb{R}$ and $\omega|_{[R,\infty)\times M} = d(e^a \lambda)$.
$\overline{W}^\infty = W \cup_{\partial W} ([R,\infty] \times M)$, $\partial \overline{W}^\infty = M$.
Choose J on W^∞ ω-compatible st $J|_{[a_0,\infty)\times M} = J_+$ for $a_0 \in [R,\infty)$.
Let $\mathcal{F}_0 = \bigcup$ leaves in $\mathcal{F}_+$ lying in $[a_0,\infty)\times M$.

Let $\mathcal{M}$ = moduli space of finite energy J-curves in W^∞, and let $\overline{\mathcal{M}}$ be a compactification of $\mathcal{M}$. Let $u_0 \in \mathcal{F}_0$ an asymptotically simple curve, and let $\mathcal{M}_0 \subseteq \mathcal{M}$ connected compact containing u_0.

Umut - High dimensional J-holomorphic curve classification of fillings

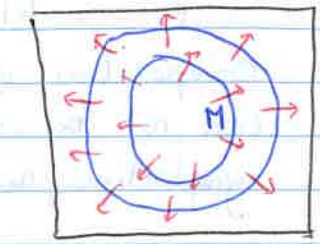
Rem: the main point here is an application of the classification of fillings.

(M^{2n+1}, λ) connected contact manifold; $(V^{2n}, \omega, \mathcal{L})$ Liouville manifold ($\mathcal{L}$ ← Liouville v.f.) that is of "finite type":

modeled after symplectization.

We say M is V-spliffable if M is contactomorphic to a convex hypersurface $\tilde{M}$ in $V \times \mathbb{C}$ such that M divides $V \times \mathbb{C}$ into 2 subsets, one of which is modeled after the positive symplectization of M, i.e. in that component we have $\mathcal{L} \neq 0$, $\phi^t_{\mathcal{L}}(M)$ is the flow under $\mathcal{L}$, and $\bigcup_{t \geq 0} \phi^t_{\mathcal{L}}(M)$ cover it.

V $= \tilde{M}$



$\mathbb{C}$ with standard radial Liouville structure

Theorem [Barth-Geiges-Zehmisch] (2016) Let W be an aspherical strong filling of M. Then, $\exists$ diagram M is V-spliffable.

$$\begin{tikzcd} H_*(V) \arrow[rr, two heads] \arrow[dr] & \circlearrowleft & H_*(W) \\ & H_*(M) \arrow[ur] & \end{tikzcd}$$

and same for π_1.

Rem: $H_*(V) \to H_*(W)$ surjective implies that $H_*(M) \to H_*(W)$ is surjective as well.

Rem 1: for the surjection statement involving M and W only, we don't need to know what V is.

Theorem [Cieliebak, Cieliebak-Eliashberg] If M admits a subcritical Weinstein filling, then it is spliffable. (subcritical $\Rightarrow$ $\mathbb{C} \times$ smthg)

Rem 2: Barth-Geiges-Zehmisch actually prove a much stronger statement about the topological classification of fillings.

⚠ In high dimension, we don't get any information about symplectomorphism type.

Corollary [Eliashberg-Floer-McDuff] Let W be an aspherical filling of (S^{2n-1}, ξ_{std}), $n \geq 3$. Then, W is diffeomorphic to $\mathbb{D}^{2n}$.

Proof: S^{2n-1} is $\mathbb{C}^{n-1}$-spliffable. So $H_*(W) = 0$ in lots of dimensions, and with $\pi_1 = 0$, this implies (by smooth topology) the statement. □

Note: BGZ proof is a careful generalization of EFM proof.

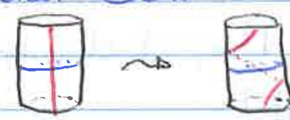
Corollary 2: let Σ^n be a closed manifold. The unit sphere bundle $(ST^*\Sigma, \xi)$ doesn't admit any subcritical filling (Weinstein)

Proof: if it did, it would give an upper bound on other fillings, which is $H_n = 0$ (middle dimension; subcritical). But it does admit the unit disk bundle as a filling. $\boxtimes$

$\rightsquigarrow V$ is $(2n-2)$-Weinstein mfld $\rightsquigarrow$ no homology. $\square$

Corollary 3: let X^{2n} be a Liouville manifold. Then, a nonempty composition of generalized Dehn twists along exact Lagrangian spheres can not be isotopic to the identity as a compactly supported symplectomorphism.

♪ Interlude ♪ on higher dimensional Dehn twists.

Model Dehn twist for $n=1$:  It is a compactly supported symplectomorphism of T^*S^1 which is non trivial, and acts as the antipodal map on $S^1 = 0$-section. (all powers are non trivial)

This generalizes to higher dimensions: $\exists\ \tau : T^*S^n \to T^*S^n$ compactly supported symplectomorphism with the same properties.

Hence, for any Lagr. embedding $S^n \xrightarrow{L} X^{2n}$ into a symplectic X^{2n}, we get $\tau_L : X^{2n} \to X^{2n}$ a compactly supported symplectomorphism by implanting τ into a Weinstein neighborhood.

Rem: construction of τ relies on the geodesic flow of S^n being periodic.

Given Lagr. embeddings $L_1, \dots, L_k$ into a Liouville domain X^{2n}, we can construct a Lefschetz fibration, with the fiber of $*$ being X^{2n}, and the vanishing cycles being the L_i's.

Y
$\downarrow$

$\mathbb{D} \simeq$  $\rightsquigarrow OB(X^{2n}, \tau_{L_1} \circ \dots \circ \tau_{L_k}) = \partial Y$ (corners smoothed).

This has a Weinstein filling with k critical handles

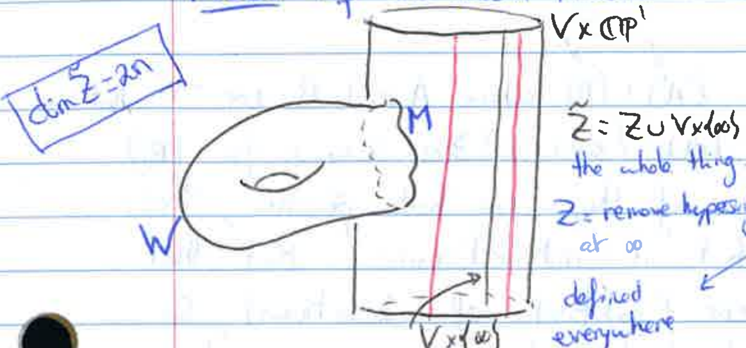
$\Rightarrow b_{n+1}(\text{filling}) \geq k - \text{constant}(X)$

♪ End of interlude ♪

<u>Proof</u>: (of ~~main theorem~~ corollary 3) Let $(M,\xi) = OB(X^{2n}, id)$; then M is X^{2n}-splittable by definition. $\Rightarrow b_{n+1}$(aspherical filling) $< C$, by the main theorem.

Assume $w := \tau_{L_1} \circ \dots \circ \tau_{L_k}$ is isotopic to the identity $\Rightarrow w^N$ also. By elementary symplectic geometry, $OB(X^{2n}, w^N) \underset{\text{contacto}}{\simeq} OB(X^{2n}, id)$. But the LHS admits fillings with an arbitrarily large b_{n+1}, as $b_{n+1} \geq Nk$ - constant by the interlude. $\square$

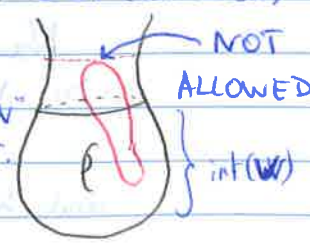
<u>Proof</u> of the main theorem: $M \hookrightarrow V \times \mathbb{D}$ ($\subseteq V \times \mathbb{C}$); can replace the interior by W (ie glue in).



Choose $\mathbb{D} \to \mathbb{CP}^1$ such that $int(\mathbb{D}) \to \mathbb{CP}^1 - \{\infty\}$.
Choose an almost-$\mathbb{C}$-str. of the form $J_{\tilde{Z}} = J_V \oplus i_{st}$ along $V \times \mathbb{CP}^1 - int(W)$.
J_V is admissible for $V \rightsquigarrow J_{\tilde{Z}}$ is also admissible for $V \times \mathbb{CP}^1$.

Fix $a, b \in \mathbb{CP}^1$, and add (in red) the hypersurfaces $V \times \{a\}$ and $V \times \{b\}$.

♪ Interlude ♪ on maximum principles.
If W is a finite type Liouville manifold $\exists$ class of a-$\mathbb{C}$-str (admissible) such that holomorphic curves can not touch levels of the convex cone. In other words: if $u : S^2 \to W$ is a holomorphic curve that is not contained in $int(W)$, then it's <u>constant</u>.



♪ End of interlude ♪

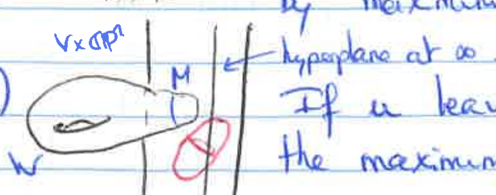
Rem: we have a max principle for V, and for $V \times \mathbb{CP}^1$. We'll use both.

//// = V looks like symplectization there.

1)

If u intersects the shaded region, then u is a branched covering of $\{v\} \times \mathbb{CP}^1$ for v is the shaded region, by maximum principle for V.

2)



If u leaves W, then it has to intersect $V \times \{\infty\}$, by the maximum principle for $V \times \mathbb{C}$.

Let $\mathcal{M}$ be the moduli space $u: \mathbb{CP}^1 \to \tilde{Z}$ such that

i) $[u] = [\{v\} \times \mathbb{CP}^1]$ for v large enough.

ii) $u(-1) \in \{a\} \times V$, $u(1) \in \{b\} \times V$, $u(\infty) \in \{\infty\} \times V$.

Rem: Every curve in $[\{v\} \times \mathbb{CP}^1]$ has to intersect all 3 $\{a\} \times V$, $\{b\} \times V$ and $\{\infty\} \times V$; the condition ii) is just a slicing condition so we don't have to mod out by anything.

Main proposition: * $\mathcal{M}$ is an oriented manifold of dim $2n-2$.

* $\mathcal{M} \times \mathbb{CP}^1 \xrightarrow{ev} \tilde{Z}$ is proper and degree 1.

Claim: $[\{v\} \times \mathbb{CP}^1]$ is simple.

Proof: assume that $[\{v\} \times \mathbb{CP}^1] = [\overset{o}{\underset{x}{A}}] + [\overset{o}{\underset{x}{B}}]$ where A and B are J-hol.
By positivity of intersections, $[A] \cdot [\{\infty\} \times V] \geq 0$ (same for $[B]$)
$\Rightarrow$ WLOG $[A] \cdot [\{\infty\} \times V] = 0$: if they were both positive, their sum would intersect twice, but it intersect once. But that intersection # is the geometric one (positivity of intersections), so it does not intersect ∞, so it has to be constant by max. principle. $\square$

Proposition: $H_*(\mathcal{M} \times \mathbb{C}) \xrightarrow{ev} H_*(Z)$ is surjective.

Proof: $ev_*(ev^!) : H_*(Z) \to H_*(Z)$ is an iso. $\square$
($ev^!$: integration along fibers)

take H_* everywhere

Now, consider the diagram

$$\begin{tikzcd}
\mathcal{M} \times \{1\} \arrow[r, "ev"] \arrow[d, hook, "\cong"'] & V \times \{a\} \arrow[r] \arrow[d, hook] & V \times \mathbb{C} - \mathrm{int}\, W \cong M \arrow[dl] \\
\mathcal{M} \times \mathbb{C} \arrow[r, two heads, "ev"', "\text{Surj by Prop}"'] & Z \cong W &
\end{tikzcd}$$

So $V \times \{a\} \to Z$ has to be surjective,

and so $\underset{\cong M}{V \times \mathbb{C} - \mathrm{int}\, W} \to \underset{\cong W}{Z}$ also.

So, $H_*(M) \to H_*(W)$ is surjective. $\square$

Rem: $ev: \mathcal{M} \times \{1\} \to V \times \{a\}$ lands in $V \times \{a\}$ by the slicing condition ii) above.

Sarah - Applications of Wendl's theorem to classification of fillings

Ref. Plamenevskaya - van Horn - Morris : Planar open books, Monodromy factorizations and Symplectic fillings.

Recall: Stein fillings $\longleftrightarrow$ Positive factorizations of monodromies of comp. OB
Problem: need to know all open books compatible.
Theorem: if (Y,ξ) admits a planar OBD, then every strong symplectic filling is symplectically deformation equivalent to a blow-up of an allowable Lefschetz fibration, compatible with the given OB.

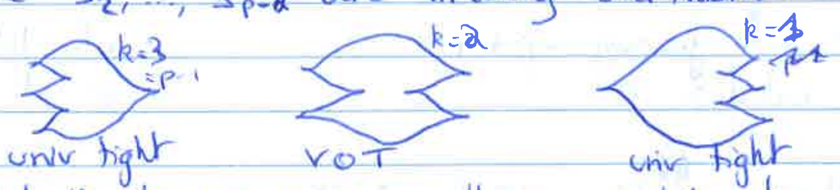
Definition: $L(p,1) :=$ quotient of $S^3 \subseteq \mathbb{C}^2$ by $\mathbb{Z}_p$-action $(z_1,z_2)\mapsto(e^{\frac{2\pi i}{p}}z_1, e^{\frac{2\pi i}{p}}z_2)$.
Definition: (Y,ξ) is overtwisted if it contains an OT disk, otherwise it's tight.
If (Y,ξ) is tight and we lift it to the universal cover $(\tilde{Y},\tilde{\xi})$ and it's tight again, then ξ is universally tight. Otherwise, it's ~~univ~~ virtually OT.
($\hookrightarrow$ tight & not univ. tight)

OT / tight / univ tight / vir. OT

Goal: Theorem: every virtually OT contact structure on $L(p,1)$ has a unique Stein filling up to symplectic deformation, which is also its unique weak filling, up to symplectic deformation and blow up.
Corollary: if $p\neq 4$, then can replace "VOT" with "tight".

[Honda 2000] The tight contact structures on $L(p,1)$ are $\xi_1,\dots,\xi_{p-1}$, where ξ_k arises from Leg. surgery on the unknot with $p-2$ stabilizations: (stab = adding handle and Dehn twist) ~~we have~~ with k cusps on the left for pos. stab
The $\xi_2,\dots,\xi_{p-2}$ are virtually overtwisted. $\rightarrow$ pos, $\rightarrow$ neg.

MB B U vK U



Total # of cusps is p. # cusps on left is k.

In the OB from Leg. surgery for ξ_k, the page is a disk with $n=p-1$ holes, and if δ_j = curve around j^{th} hole, the monodromy is
$\Phi = D_\alpha D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n} D_\beta$, where α contains the holes 1 through k, and β contains holes k through n.
$\Rightarrow$ Stein filling: $D^4 \cup$ 2-handles

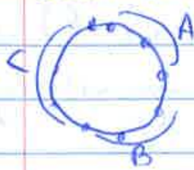
Rem. there is a procedure for going to OB to surgery diagrams, so, here, we can see these holes as the stabilizations:

see lemma 2 ← $D_\alpha D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n}$ corresponds to S^3, and adding the last D_β corresponds to doing the surgery.

here we use $k \neq 1$ and $k \neq p-1$, so need VOT. Also, goes bad when $p=4$.

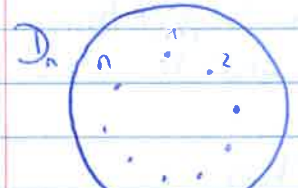
Lemma 1: any positive factorization of Φ takes the form $D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n} D_{\beta'}$, where α' and β' enclose the same loops as α and β.

Margalit/McCammond 2009: if D_n is the disk with n holes, $MCG(D_n)$ has a presentation with generators all convex Dehn twists and relations (i) Dehn twist along disjoint curves commute



(ii) Lantern relations: $D_A D_B D_C D_{A\cup B\cup C} = D_{A\cup B} D_{A\cup C} D_{B\cup C}$ if A, B, C are ~~orient~~ ordered clockwise.

⚠ maybe $A\cup B\cup C \neq$ all the holes



γ a simple closed curve is convex if it is isotopic to the boundary of the convex hull of some set of holes.

If γ is convex, D_γ factors into a product of ∂ and pairwise Dehn twists, using the lantern relations

around a hole: δ_i; around 2 holes:

Lemma 2: if $\Psi = D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n}$, then the OB with monodromy Ψ represents (S^3, ξ_{std}). The knot in S^3 induced by β' is the unknot with framing $-p+1$

ie include the page D_n inside S^3, and that's the knot

Proof: Kirby calculus. □

Proof of theorem: X is LF for $D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n} D_{\beta'}$, then by Lemma 2, it is diffeomorphic to D^4 with a 2-handle attached along the unknot coming from β' with framing $-p+1$. X has a Stein structure, coming from surgery on a knot with $tb = -p+1$. It's also (apparently) easy to compute the rotation number, so we know the knot, and it turns out to be

This is the only way to get $(L(p,1), \xi_k)$. The compatible symplectic structure is unique up to symplectic deformation, by Gompf 2004.
$\Rightarrow$ All Stein structures are the same.
$\Rightarrow$ Stein fillings are unique, by Wendl: we don't have to use any other OBD; it's enough to do it with this one.

So: by Wendl's theorem, we know there is a unique strong filling (which happens here to be Stein).

To prove the statement about weak fillings in the theorem, we invoke the following theorem:

Theorem [Ohta-Ono 1999] any weak filling of a rational homology sphere can be turned into a strong filling.

$\square$

Kevin - Introduction to Seiberg-Witten invariants

Ref: Hutchings-Taubes: An introduction to the SW equations on symplectic manifolds (§2-3).

SW-equations: $\begin{cases} D_A \Psi = 0 \\ F_A^+ = q(\Psi) + i\mu . \end{cases}$, on a 4-manifold

Here, $(A, \Psi) \in \underset{\text{connection}}{\mathrm{Conn}(L)} \times \underset{\text{spinor}}{\Gamma(S_+)}$. And $\mu \in C^\infty(\Lambda^2_+ T^* X)$.

$\boxed{\triangle \text{ Equations in } (A,\Psi) \text{ depending on metric } g, \text{ Spin}^{\mathbb{C}}\text{-structure and } \mu.}$

1) Spin$^{\mathbb{C}}$-structures and associated bundles:

For $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$, so let $Spin(n)$ be the connected double cover. Define also $Spin^{\mathbb{C}}(n) = (Spin(n) \times U(1))/\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $Spin(n)$ by Deck transformations and on $U(1)$ by negation.

For $(X^n, g = \text{metric})$, let $\begin{matrix} Fr \\ \downarrow \\ X \end{matrix}$ such that the fiber over $x \in X$ is $\mathrm{Isom}(\mathbb{R}^n, T_x X)$; it is an $SO(n)$-bundle. Can we lift it to $Spin^{\mathbb{C}}(n)$? A lift is a $Spin^{\mathbb{C}}$-structure.

$$\begin{tikzcd} \hat{F} \times Spin^{\mathbb{C}}(n) \arrow[r] \arrow[d] & \hat{F} \arrow[d] \\ Fr \times SO(n) \arrow[r] & Fr \arrow[d] \\ & X \end{tikzcd}$$

Theorem: any X^4 oriented has a spin$^{\mathbb{C}}$-structure (an affine space modelled on $H^2(X;\mathbb{Z})$ worth of it, actually)

Associated bundles: for $\rho : G \to \mathrm{Aut}(V)$ and $\begin{matrix} G \to P \\ \downarrow \\ X \end{matrix}$, we can form the vector bundle $\begin{matrix} V \to P \times_\rho V \\ \downarrow \\ X \end{matrix}$.

Get 2 $\mathbb{C}^2$-bundles $S_\pm$ from $Spin^{\mathbb{C}}(4) \xrightarrow{S_\pm} U(2)$ and a line-bundle L from $Spin^{\mathbb{C}}(4) \xrightarrow{k} U(1)$, by taking $V = \mathbb{C}^2$ and $\mathbb{C}$ respectively.

We have $SO(4) \cong SU(2) \times SU(2)/\pm 1$. Indeed, write $\mathbb{R}^4 = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| \ a, b \in \mathbb{C} = \mathbb{R}^2 \right\}$, and $(h_-, h_+) \in SU(2) \times SU(2)$ acts by $(h_-, h_+).x = h_- x h_+^{-1}$.

So also have $Spin(4) = SU(2) \times SU(2)$
and $Spin^{\mathbb{C}}(4) = SU(2) \times SU(2) \times U(1)/\pm 1 \ni [h_-, h_+, \lambda]$

Define $k([h_-, h_+, \lambda]) = \lambda^2$.

Also $U(2) = SU(2) \times U(1)/\pm 1$, and so we define
$S_+([h_-, h_+, \lambda]) = [h_+, \lambda]$ and $S_-([h_-, \lambda])$.

Definition: $S_\pm$ are spinor bundles, and L is the determinant line bundle of S_+ (or S_-; they are equal): $\det S_\pm = \Lambda^2 S_\pm$.

Clifford multiplication: map $cl : T^*X \to End(S_+ \oplus S_-)$.
Properties: * $cl(v)$ interchanges S_+ and S_-, $\forall v$: $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.
* $cl(v)^2 = -|v|^2$
* $|v| = 1 \Rightarrow cl(v)$ is unitary.

Construction: want $cl : T^*X \otimes S_+ \to S_-$ (and same swapping S_+ and S_-).
At a point $\mathbb{R}^4 \times \mathbb{C}^2 \to \mathbb{C}^2 : (x, \psi) \mapsto x\psi$.

Suppose we have $[h_-, h_+, \lambda] \in Spin^{\mathbb{C}}(4)$ ($x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, ψ); it acts on x as $h_- x h_+^{-1}$ and on ψ as $\lambda h_+ \psi$ and we have $h_- x h_+^{-1} \cdot \lambda h_+ \psi = \lambda h_- x \psi$, so we see that the representations match.
Also, define the other part of it: $T^*X \otimes S_- \to S_+ : (x, \psi) \mapsto -\bar{x}^t \psi$.

Given a $Spin^{\mathbb{C}}(4)$-structure $\hat{F} \to X$, we want a connection, ie a $\mathcal{L}ie(Spin^{\mathbb{C}}(4))$-valued 1-form on $\hat{F}$ satisfying stuff. We have $\mathcal{L}ie(Spin^{\mathbb{C}}(4)) = \mathcal{L}ie(SO(4)) \oplus \mathcal{L}ie(U(1))$ (since we took covers to define $Spin^{\mathbb{C}}(4)$).

Levi-Civita lands here, and a connection on L lands there.
Rem: for A on vector bundle E, get $\nabla_A : C^\infty(E) \to C^\infty(T^*X \otimes E)$.
We want "compatibility" with the metric: $dg(x,y) = g(\nabla_A x, y) + g(x, \nabla_A y)$, and ∇_A should be "torsion-free".

Definition: Given a metric-compatible connection A on L, let D_A to be the composition $C^\infty(S_\pm) \xrightarrow{\nabla_A} C^\infty(T^*X \otimes S_\pm) \xrightarrow{cl} C^\infty(S_\mp)$.
This explains the 1st of the SW-equations.

Also, A gives a curvature $F_A \in C^\infty(i\Lambda^2 T^*X)$. A metric gives a Hodge $*$, which in turns gives eigenspaces
$$\Lambda^2 T^*X = \underbrace{\Lambda^2_+ T^*X}_{} \oplus \Lambda^2_- T^*X ;$$
Definition: F_A^+ is the part living here.

We extend $cl: \Lambda^2 T^*X \to End(S_\pm): v \wedge w \mapsto \frac{1}{2}[cl(v), cl(w)]$.
In particular, restricting to the positive part:
$cl_+ : \Lambda^2_+ T^*X \otimes \mathbb{C} \to End(S_+)$, and we define
Definition: $q(\Psi) = cl_+^* (\Psi \otimes \Psi^*)$.

Rem: on a symplectic manifold, there is a natural $spin^c$-structure where the SW-equations simplify to something easier.

2) What is the invariant?

Idea: $\mathcal{F}(A,\Psi) = \begin{pmatrix} D_A\Psi \\ F_A^+ - q(\Psi) - i\mu \end{pmatrix}$; we want to study $m_{g,s,\mu} := \{\mathcal{F}(A,\Psi) = 0\}$. Let $\mathcal{M} = m/\mathcal{G}$; it is a nice(-ish) manifold, from which we can extract homological invariants.
Rem: think of μ as a perturbation.

What is the gauge-group $\mathcal{G}$? Let $\mathcal{G} = C^\infty(X, S^1)$, and it acts as $u\cdot(A,\Psi) = (A^2 - 2u^{-1}du, u\Psi)$. We can check that the SW-equations are invariant under that.
This action is free, except if $\Psi = 0$ where we get a S^1-stabilizer: u being constant. Definition: (reducible)

So $\mathcal{M} = m/\mathcal{G}$, and let $\mathcal{M}^\circ = m/\mathcal{G}^\circ$, where $\mathcal{G}^\circ$ is: pick $* \in X$, and let $\mathcal{G}^\circ = \{\phi \in \mathcal{G} \mid \phi(*) = 1\}$. The idea is that this way, u can not be constant (unless it's $u \equiv 1$), so we don't have S^1-stabilizers as above $\Rightarrow$ free action.

So, $\mathcal{M}^0$ has more chances of being non-singular than $\mathcal{M}$.

Theorem: fix s, g.
(a) $\mathcal{M}$ is compact
(b) For generic μ, $\mathcal{M}^0$ is a smooth finite dimensional manifold with smooth S^1-action. If $b_{2,+} > 0$, then for generic μ, $\mathcal{M}$ is smooth and $\mathcal{M}^0 \to \mathcal{M}$ is a principal S^1-bundle.
(c) For generic μ, $2d := \dim \mathcal{M}$ is a purely topological quantity (only involving X and $c_1(L)$), that vanishes for almost-$\mathbb{C}$-structures:
$= b_1 - 1 - b_{2,+} + \frac{1}{4}(c_1(L) \cdot c_1(L) - \tau)$
$\tau = b_{2,+} - b_{2,-}$.
$c_1^2 - 3\,\mathrm{sign} - 2\chi$: this is 0 if manifold is almost complex.
(d) Orientations on $\mathcal{M} \longleftrightarrow$ vector space orientations on $H^0(X;\mathbb{R}) \oplus H^1 \oplus H^2_+$.
(e) A generic path (g_t, μ_t) gives a smooth compact oriented cobordism on $\mathcal{M}^0$. Same on $\mathcal{M}$ if $b_{2,+} > 1$.

Rem: (b)-(e) are standard index elliptic theory results (once you realize that reducible solutions are codimension $b_{2,+}$ in μ). (ie there is a codim $b_{2,+}$ space of μ's that give reducible solutions = bad points)

Theorem: * $b_{2,+} > 1 \Rightarrow SW_X$ is a diffeomorphism invariant.
Definition: $SW_X : \mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{Z}$ is defined:
* 0 if $b_{2,+} - b_1$ is even
* 0 if $d < 0$
* $d = 0$: $SW_X(s) = \sum \pm 1$ (count $\mathcal{M}$)
* $d > 0$: $\int_{\mathcal{M}} e^d$, $e \in H^2(\mathcal{M};\mathbb{Z})$.

Rem: it's conjectured that this is 0 if $d > 0$. And proved for symplectic mflds.
Theorem continued: * SW_X is finitely supported
* $SW_X \longleftrightarrow SW_{X \# \overline{\mathbb{CP}^2}}$ contain the same information ($\exists$ blow up formula)
* $X = Y \# Z$, $b_{2,+}(Y, Z) > 0 \Rightarrow SW_X = 0$.
* $s \mapsto \bar{s}$ has for effect $SW(\bar{s}) = \pm SW(s)$.

Bochner-Wietzenböck formula: $D_A^* D_A = \nabla_A^* \nabla_A + \underset{\text{scalar curvature}}{\frac{s}{4}} + \frac{1}{2} cl(F_A)$.
Lemma: if $F(A,\psi) = 0$, $\psi \neq 0$, then $|\psi|^2 \leq \max_X (-s/2 + 2|\mu|)$.
$\Rightarrow$ if $s > 0$, for small μ, there is no solution.
This gives regularity for $F \rightsquigarrow$ Ascoli-Arzela $\rightsquigarrow$ compactness.

Jie - Symplectic Kodaira dimension 0

Actually: just some characterization of $\mathbb{Q}H_*(K3)$

Main theorem: symplectic 4-mfld, with $c_1=0$, $b_1=0$, $b_2^+>1$.
Then, it's a $\mathbb{Q}H_*(K3)$.
Rem: still true if c_1 torsion.

Theorem A: [Taubes] X symplectic with $b_2^+>1$, then $SW_X(K_\omega)=\pm 1$.
Theorem B: [Morgan - Szabo] if X spin with $b_1=0$, $b_2^+=4n-1$, $b_2^-=20n-1$, $n>1$, then $SW(0)=0 \mod 2$.
↳ trivial spin$^{\mathbb{C}}$ structure.

Here, $K_\omega = \Lambda^{2,0}TM \rightsquigarrow c_1 = c_1(TM) = c_1(\Lambda^{2,0}TM) = c_1(K_\omega)$, for ω a symplectic form.
0 is the trivial spin$^{\mathbb{C}}$-structure: $\det(S_+)=$ trivial, $c_1=0$.
Spin$^{\mathbb{C}}$ on $X \longleftrightarrow H^2(X;\mathbb{Z})$: every 2 spin$^{\mathbb{C}}$-structures differ by tensoring a line-bundle, and that difference is c_1.

Proof of main theorem
$b_1=0 \Rightarrow \chi = b_2^+ + b_2^- + 2$ by Poincaré duality.
$c_1^2(K_\omega) = 3\sigma + 2\chi$ by Hirzebruch signature theorem,
so $c_1(K_\omega)=0 \Rightarrow 3\sigma+2\chi=0$.
Also, $c_1(TM)=0 \Rightarrow w_2(TM) = c_1(TM) \mod 2 = 0$, so M is spin (w_2 = obstruction to being spin).
By Rochlin's theorem: $16 \mid \sigma$, so $\sigma = b_2^+ - b_2^- = 16n$.
$\Rightarrow b_2^+ = 4n-1$, $b_2^- = 20n-1$.

By theorem B, if $n>1$, then $SW(0)=0 \mod 2$. But the manifold is symplectic, so by theorem A, $SW(K_\omega)=\pm 1$.
So we have $n=1$, otherwise there is a contradiction.
$\Rightarrow b_1=0$, $b_2^+=3$, $b_2^-=19$.

M spin $\Rightarrow$ intersection form Q_M is even $\Rightarrow Q_M = 2E_8 \oplus 3H$,
so we have a $\mathbb{Q}H_*(K3)$. $\square$
Rem: for background, see chapter 1 of Gompf - Stipsicz.

Recall classification of intersection forms: we say Q is even if $Q(a,a) \equiv 0 \mod 2$ for all $a \in H^2(X;\mathbb{Z})$. ~~For Q~~ We have

$$Q \begin{cases} \text{even: } mE_8 \oplus nH, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \text{odd: } b_2^+\langle 1\rangle \oplus b_2^- \langle -1 \rangle \end{cases}, \text{ for } Q \text{ indefinite.}$$

m and n are linear functions of Betti numbers: E_8 are all negative, and H has 1 pos and 1 neg.

Proof of theorem A:

We need to know what the canonical spin$^{\mathbb{C}}$-structure on (X,ω) is.

(Choose J s.t. ω is self-dual). $\Lambda^{0,*}T^*X$ is spin$^{\mathbb{C}}$-bundle, and it splits as $\Lambda^{0,*}T^*X = \underbrace{(\Lambda^{0,0} \oplus \Lambda^{0,2})}_{=S_+} \oplus \underbrace{\Lambda^{0,1}}_{=S_-}$.

The Clifford multiplication of $T^*X \otimes \mathbb{C}$ on $\Lambda^{0,*}T^*X$ is given by [pick g such that ω is self-dual, and then take the associated J]

$cl(v).\alpha = \sqrt{2}\,(v^{0,1} \wedge \alpha - \iota(\overline{v^{1,0}})\,\alpha) \qquad v \in T^*X \otimes \mathbb{C},\ \alpha \in \Lambda^{0,*}T^*X$

We have $S_+ = \underline{\mathbb{C}} \oplus K^{-1}$, and $S_- = TX$. For S_+, this is the splitting of $cl(\omega)$ into $(-2i)$-eigenspace and $(2i)$-eigenspace.

There is a canonical inclusion $U(2) \hookrightarrow Spin^{\mathbb{C}}(4)$, for A a transition matrix for the manifold

$$\begin{tikzcd} U(2) \arrow[r, hook] & Spin^{\mathbb{C}}(4) \arrow[dl, "S_+"'] \arrow[dr, "S_-"] & \\ U(2) & & U(2) \end{tikzcd}$$

$A \mapsto \begin{pmatrix} 1 & \\ & \det A \end{pmatrix}$, $\quad A \mapsto A$

Any $\psi \in \Gamma(S_+)$ is (α, β) where $\alpha \in \Gamma(\underline{\mathbb{C}}) = C^\infty(X)$, $\beta \in \Gamma(K^{-1})$.
Now, the $q(\psi)$ term in the SW-equations has a better form: the equation becomes $F_A^+ = \underbrace{i(|\alpha|^2 - |\beta|^2)\,\omega + 2(\bar{\alpha}\beta - \alpha\bar{\beta})}_{q(\psi)} + i\mu$.

And in this case, $\dim \mathcal{M} = ad = 6 = \frac{1}{4}(c_1^2(K_\omega) - (3\sigma + 2\chi)) = 0$

Take $u_0 = 1 \in \Gamma(S_+)$; $\exists$ spin$^{\mathbb{C}}$ connexion A_0 such that $\nabla_{A_0} u_0 \in \Omega^1(X, K^{-1})$. Indeed: choose any spin$^{\mathbb{C}}$ connexion A; any other will be $A + a$ for $a \in \Omega^1(i\mathbb{R}) = \mathrm{Lie}\, U(1)$; we have $\nabla_{A+a} u_0 = \nabla_A u_0 + \frac{1}{2} a u_0$ ← Clifford multiplication. Take $a = (-2\nabla_A u_0)_{\underline{\mathbb{C}}}$ (projection), and $A_0 = A + a$: this way, $\nabla_{A_0} u_0$ sits in $\Omega^1(X, K^{-1})$.
And we have: $A_0 \rightsquigarrow D_{A_0} u_0 = 0$.

Lemma: $D_{A_0+2a}\binom{\alpha}{\beta} = \bar\partial_a \alpha + \bar\partial_a^* \beta$, where $\bar\partial_a = (\nabla_a)^{0,1}$.

With this, the SW-equations become $\begin{cases} \bar\partial_a \alpha = -\bar\partial_a^* \beta \\ F_A^+ = i(|\alpha|^2 - |\beta|^2)\omega + 2(\bar\alpha\beta - \alpha\bar\beta) + i\mu \end{cases}$.

Now, we will use a specific perturbation term: take $\mu = -r\omega - iF_{A_0}^+$, where $r \in \mathbb{R}_+$. Then, the equations become $\begin{cases} (da)^{0,2} = r\bar\alpha\beta \\ \langle \omega, da\rangle = ir(1 - |\alpha|^2 + |\beta|^2) \end{cases}$, where $A = A_0 + a$.

Rescaling $\psi = \sqrt{r}(\alpha, \beta)$, we get the inequality

$$\int \left(\left(1 - \frac{2c}{r}\right) |\nabla_a \alpha|^2 + r(1 - |\alpha|^2)^2 \right) \leq \underbrace{2\pi [\omega] \cdot c_1}_{\text{Chern-Weil theory}} \overset{(\mathfrak{C})}{=} 0$$

Taking r large enough: need $|\alpha| = 1$, $\nabla_a \alpha = 0$.
So $\alpha \equiv 1$, $\beta \equiv 0$, $a = 0$ is the unique solution, up to gauge transformation.
This proves the Theorem A: for any other choice of μ, get something cobordant to 1 point $\Rightarrow$ sum of points is ± 1. $\square$

Proof of theorem B:

This part uses the $Pin(2)$ action on $\mathcal{M}$, where $Pin(2)$ is the subgroup of $Sp(1)$ generated by j and S^1, where $j \circlearrowright \mathbb{C}^2$ by $(z_1, z_2) \mapsto (-\bar z_2, \bar z_1)$ (i.e. quaternion multiplication on the right).

Recall $\mathcal{F}: \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-) : (a, \psi) \mapsto \begin{cases} D_{A_0+a}\psi \\ F^+_{A_0+a} - q(\psi) + i\mu \end{cases}$

There is a $Pin(2) \circlearrowright \mathcal{M}_0$, and $\mathcal{M} = \mathcal{M}_0 / S^1$.
Since $Pin(2)$ is gen. by j and S^1, j acts on $\mathcal{M}$.

$(A_0, 0)$ is one reducible solution. Away from there, j is a fixed point-free involution, so the solutions in $\mathcal{M} - (A_0, 0)$ come in pairs.
Near $(A_0, 0)$, we have $L = d\mathcal{F}_{(A_0,0)} : \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-)$, and we have the local model $Q: \operatorname{Ker}(L) \to \operatorname{coker}(L)$; for $P: \ker D_{A_0} \to \operatorname{coker} D_{A_0}$

$\operatorname{ind} D_{A_0} = -\frac{5}{4}$ "Kuranishi map" $b_2^{+} \quad \operatorname{Ker}(L) = \operatorname{Ker} D_{A_0} \quad \operatorname{coker}(L) = \operatorname{Coker} D_{A_0} \oplus \mathcal{H}^2_+$

get $Q|_{\operatorname{Ker} P} : \mathbb{H}^{n} \to \mathbb{R}^{4n-1}$. For $n = 1$, $\dim \ker = 1$, so $\ker/S^1 = \pm 1$. But if $n > 1$, $\dim \operatorname{Ker} > 1$, so it gives many directions to deform the solutions. In any dir, can go in $+$-dir or $-$-dir, so they come in pairs, hence $SW(\mathfrak{o}) = 0 \bmod 2$. $\square$

Tom - Fillings of unit cotangent bundles

yes for $g=0$ or 1; unknown for $g\geq 2$.

There are ∞^{ly} many strong fillings of those. Is there a unique exact one? There is clearly one: the unit disk bundle.

Theorem 2: if (Y_g, ξ_g) is the unit cotangent bundle of Σ_g, then the homology of any exact filling is that of the unit disk cotangent bundle.

Theorem 1: if (Y, ξ) admits some Calabi-Yau cap, then $|\{(b_1(X), b_2(X), b_3(X)) \mid X \text{ exact filling of } Y\}| < \infty$.

Theorem 3: there is a unique Stein filling of (Y_g, ξ_g), up to s-cobordism rel boundary.

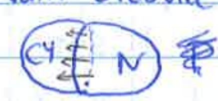
Ref for 1 & 2: "Calabi-Yau caps, Uniruled caps and symplectic fillings" by Li, Max, Yasui (=LMY).

Definition: a Calabi-Yau cap of a contact 3-manifold is a compact (P, ω) which is a strong concave filling with $c_1(P)$ torsion.

Proposition: the pairing $H^2_{dR}(X;\mathbb{R}) \times H^2_{dR}(X, \partial X; \mathbb{R})$: $([A], [(B,b)]) \mapsto \int_X A\wedge B - \int_{\partial X} A \wedge b$ is well defined.

near boundary

Proof of theorem 1: pick a Calabi-Yau cap with Liouville contact form, and let (N, ω_N) be an exact filling:



Together, they form a closed 4-manifold.
Scaling our Liouville contact form, we can take $c_1(P)\cdot[(\omega_N, \alpha_N)] = 0$

and also sympl form

Lemma: [LMY] if (W_i, ω_i) are ~~Calabi-Yau caps~~ exact fillings of (Y, ξ) with Liouville 1-form α, then for $t \gg 0$, there is a symplectic form ω on the glued manifold with ~~$c_1(X)\cdot\omega = c_1(N)\cdot[(\omega_N,\alpha_N)] + tc_1(P)\cdot[(\omega_P,\alpha_P)]$~~

$$c_1(X)\cdot\omega = c_1(W_1)\cdot[(W_1,\alpha_1)] + t\,c_1(W_2)[(W_2,\alpha_2)]$$

(W_1, ω_1) is a symplectic cap, (W_2, ω_2) is an exact filling. (or opposite)

So for us: $c_1(X)\cdot\omega = t\,c_1(N)\cdot[(\omega_N, \alpha_N)] + c_1(P)\cdot[(\omega_P, \alpha_P)]$

Theorem: if (X, ω) is a minimal symplectic Calabi-Yau, its rational homology is that of a K3 surface, the Enriques surface, or a torus bundle over a torus.

· This concludes the proof of theorem 1, at least in some cases. $\square$

<u>Proof of theorem 2</u>: let $\mathcal{U}$ be the unit cotangent disk bundle.
<u>Lemma</u>: there is a symplectic K3 surface X which contains g distinct Lagrangian tori (all in the same homology class, and don't intersect) and a Lagrangian sphere which intersects each torus transversely at one point, ~~where X has the same homology as an exact filling.~~

Let L be the union of the tori and the sphere, glued together using Lagrangian surgery. It turns out that L is a genus g surface. Identify $\mathcal{U}$ with a tubular neighbourhood of L.
Then, $P = X - \mathrm{int}(\mathcal{U})$ is a Calabi-Yau cap. Try to conclude. $\square$

<u>Ref for 3</u>: "Fillings of unit cotangent bundles" by Sivek and van Horn-Morris.
We will focus on the following subtheorem:
<u>Theorem</u>: if (W, J) is a Stein filling of (Y_g, ξ_g), then $\underset{G}{\overset{}{\pi_1(W)}} \cong \pi_1(\Sigma_g)$.

Proof <u>Fact</u>: $\pi_1(Y_g) \simeq \langle a_i, b_i, t \mid \prod [a_i, b_i] = t^{2g-2}, [a_i, t] = [b_i, t] = 1 \rangle$;
this use the LES for fibers.
$\Rightarrow i_* : \pi_1(Y_g) \twoheadrightarrow \pi_1(W)$, since Stein: only 0, 1 and 2-handles.
Let $H \trianglelefteq \pi_1(Y_g)$ where $H = \langle a_i, b_i \rangle$. <u>Claim</u>: $i_* : H \twoheadrightarrow \pi_1(W)$.
<u>Proof</u>: $[\pi_1(Y_g) : H] = $ [illegible] $2g-2$ [illegible]
So, $[\pi_1(W) : i_*(H)] =: k \in \mathbb{N}$, $k \leq 2g-2$
"$\Rightarrow$" $(k-1)(2-2g) \geq -1$ by covering spaces theory
$\Rightarrow k = 1$, as $2-2g < 0$ by assumption. So, $i_* H \twoheadrightarrow \pi_1(W)$. $\square$

<u>Proposition</u>: $\exists$ SES $1 \to \langle t \rangle \hookrightarrow \pi_1(W) \twoheadrightarrow \pi_1(\Sigma_g) \to 1$. (as above)
<u>Proof</u>: surface groups are RFRS $\Rightarrow \exists$ $G_0 := \pi_1(\Sigma_g) \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots$, with quotients cyclic, and $\bigcap G_i = \{id\}$. Then use LHSerre SS and group cohomology, and <u>Lemma</u>: $H_2(\pi_1(W); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{2g}$.
So $n = 1$, so $\langle t \rangle = \{id\}$. $\square$

Dani - Tight contact structures and Seiberg-Witten equations

Plan: 1) The goal
2) The construction
3) Some algebraic topology
4) Some Stein geometry
5) Some Seiberg-Witten theory
6) The proof.

1) Goal: distinguish contact structures on M^3 closed. Two contact structures ξ_1, ξ_2 are homotopic if the underlying plane fields are. They are isotopic if they ~~are~~ can be joined by a path ξ_t of actual contact structures. They are isomorphic if (M^3, ξ_1) and (M^3, ξ_2) are contactomorphic. We will show: homotopic $\not\Rightarrow$ isomorphic.
Rem: isomorphic $\not\Rightarrow$ homotopic: the diffeo need to be isotopic to the identity.
Rem: isomorphic + homotopic $\not\Rightarrow$ isotopic
Rem: isotopic $\Rightarrow$ homotopic + isomorphic.
Rem: $\mathrm{Cont}(M) = \mathrm{Tight}(M) \amalg \mathrm{OT}(M)$; OT means: contains (OT disk).
Theorem: for $\xi_1, \xi_2 \in \mathrm{OT}(M)$, homotopic $\Leftrightarrow$ isotopic.
Rem: Gray $\Rightarrow$ if ξ_t family of contact structures, $\exists f_t \in \mathrm{Diff} : f_t^* \xi_t = \xi_1$.

Theorem: [Lisca-Matic] $\forall n \geq 0$, $\exists$ homology sphere with at least n contact structures which are homotopic but not isomorphic.

We will build them as different Leg. realizations of the same Kirby diagram.
We'll assume $c_1(X_1) \neq c_1(X_2)$ but X_1, X_2 with same boundary; SW will imply they have the same c_1 $\rightsquigarrow$ contradiction.

2) $K \hookrightarrow (S^3, \xi_{std})$ Legendrian.
$\rightarrow$ $tb(K) = \# K' \cap S$, where K' = Reeb pushoff and S = Seifert surface.
$\rightarrow$ $r(K)$ = rot# of TK wrt a trivialization of $\xi|_S$.
Rem: $tb(K) = \# \text{(crossing)} + \# \text{(crossing)} - \# \text{(crossing)} - \# \text{(crossing)} - \# \text{(cusp)}$ (inv. of framed knots)
$r(K) = \frac{1}{2}(\# \text{(cusp)} + \# \text{(cusp)} - \# \text{(cusp)} - \# \text{(cusp)})$ (inv. of never vertical knots)
in the front projection.

Let $L = \amalg_i K_i \subseteq S^3$ and surgery $\rightsquigarrow W = B^4 \cup_i H_i^2$ has a Stein structure (framing on K_i is $tb - 1$). Furthermore,
$\langle c_1(W), [H_i^2] \rangle = r(K_i)$: think of c_1 as trivialization to

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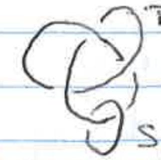
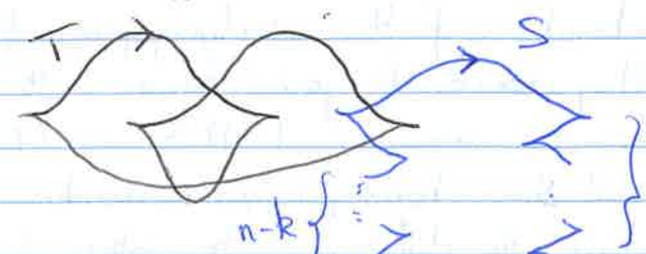
extending over the handle, this is also what rot. is.

Let $N_n = B^4 \cup_i H_i$,



, turns out that $\partial N_n = \underset{\text{Brieskorn sphere}}{\Sigma(2,3,6n-1)} = \{x^2+y^3+z^{6n-1}=0\} \cap S^5 \subseteq \mathbb{C}^3$.

We have W_n^k Stein structure on N_n for each $1 \le k \le n-1$:



$tb(S) = -n+1$
$r(S) = 2k-n$
$tb(T) = -1$
$r(T) = 0$

Rem: By Eliashberg - Fraser, this realizes all unknots with $tb = -n+1$.

Also, $c_1(W_n^k) = (2k-n)\,PD(T)$. Indeed, we have
$\langle c_1(W)$, handle corresp. to $T \rangle = 0$
$\langle c_1(W)$, handle corresp. to $S \rangle = 2k-n$
And $H^2(W_n^k)$ is free of rk 2 gen. by T and S, so we must have.

So, we have ξ_n^k contact structures on $\partial W_n^k \overset{\text{topologically}}{=} \partial N_n$.

3) Lemma (Gompf): X_i^4 are integral homology spheres almost-$\mathbb{C}$ manifolds with boundaries ∂X_1 and ∂X_2 diffeomorphic. Let ξ_i be the plane field on $T\partial X_i$ formed by the complex tangencies $\xi_i := T\partial X_i \cap J_i(T\partial X_i) \subseteq T\partial X_i$. (these are just plane fields, maybe not contact). Then,

ξ_1 and ξ_2 are homotopic as plane fields

$$\Updownarrow$$

$$c_1^2(X_1) - 3\sigma(X_1) - 2\chi(X_1) = c_1^2(X_2) - 2\chi(X_2) - 3\sigma(X_2).$$

"Proof:" compute difference between ξ_1 and ξ_2 in terms of clutching function, then in terms of Pontrjagin numbers, then use Hirzebruch signature formula. "$\square$"

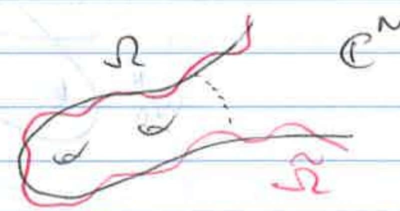
Corollary: $\xi_n^k \simeq \xi_n^{k'}$ as plane fields.

same σ and χ.

<u>Proof</u>: $X_1 \simeq X_2$ as smooth manifolds. Need to check $c_1^2(W_1^k) = c_1^2(W_n^k)$, but $c_1(W_n^k) = (2k-n)\,PD(T)$, and $PD(T)\cdot PD(T) = 0$ because that is the framing of T. □

4) <u>Ref</u>: hot tub seminar. Compact manifolds have no non-cst hol. function; Stein manifolds have plenty.
Stein: convexity $\Rightarrow$ $\bar\partial$ equation is solvable $\Rightarrow$ find lots of hol. fcts $\Rightarrow$ can separate points $\Rightarrow$ can find proper hol. embedding $\Omega \to \mathbb{C}^N$.
A Stein domain is "the compact part of a Stein manifold.
<u>Lemma</u>: let Ω be a Stein domain. Then, Ω admits a holomorphic embedding into a compact Kähler surface S satisfying

(1) minimal	(2) general type	(3) $b_+^2(S) > 1$.
no $E \subseteq S$ with $E \simeq \mathbb{CP}^1$ and $E\cdot E = -1$, ie we can't blow S down.	Kodaira dim $=2$ (maximal). This is a generic condition (think of it as neg. curvature)	b_+^2 = pos. definite part of H^2, wrt the intersection form.



<u>Rem</u>: general type $\Rightarrow$ $K\cdot[\omega] > 0$ and $K\cdot K > 0$.
<u>Proof</u>: embed it in $\mathbb{C}^N$; approximate it by one given by polynomials (Stein $\Rightarrow$ we can): $\tilde\Omega$ algebraic. Let $S = \hat{\tilde\Omega} \subseteq \mathbb{CP}^N$ the compactification given by homogeneizing the polynomials cutting out $\tilde\Omega$; then, resolve singularities. By adjunction, if the ~~power~~ degrees of polynomials are high, then S has ample canonical bundle $\Rightarrow$ minimal of general type.
For (3), we modify it slightly: take original Ω; if b_2^+ is not large enough, do a surgery, and repeat the above. To see that the pos. hom. class survives: attach N_n above to Ω; get something separated by hom. sphere $\Rightarrow$ homology splits $\Rightarrow$ pos. class survives. Also, $b_+^2(N_n) = 1$, hence it increases b_+^2. □

5) <u>Recall</u>: $SW: \mathrm{spin}^c(M^4) \to \mathbb{Z}$: count solutions to SW equations.
Call $c \in H^2(M)$ <u>basic</u> if $\exists\, \mathfrak{s} \in \mathrm{spin}^c$ st $SW(\mathfrak{s}) \neq 0$ and $c_1(\mathfrak{s}) = c$.
If $b_+^2 > 1$, $\mathrm{Basic}(M)$ is a diffeo invariant of M.
We saw earlier if M symplectic, then $\pm c_1(M) \in \mathrm{Basic}(M)$ (we saw that SW are ± 1). There could be others, <u>but</u>:

Fact: if M is compact Kähler minimal of general type, then $\pm c_1(M)$ is the only basic class.

Ingredients to prove this: let $K = \Lambda^2 T^* M = -c_1(M)$

(1) minimal + general type $\Rightarrow K\cdot K > 0$ and $K \cdot [\omega] > 0$. (1) from before: have a-$\mathbb{C}$ mfld

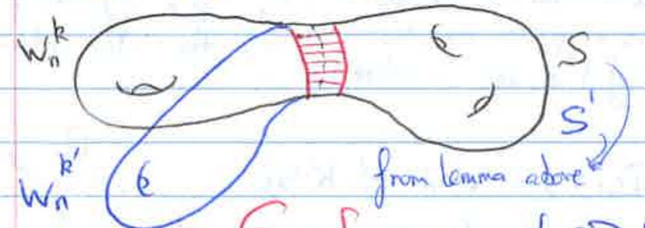
(2) assume $c \in \mathrm{Basic}(M)$; then $\underbrace{\dim(\mathcal{M}_c)}_{= c_1^2 - 2\chi - 3\sigma} \geq 0 = \underbrace{\dim(\mathcal{M}_\omega)}_{= K^2 - 2\chi - 3\sigma}$ since $SW(c) \neq 0$

$\Rightarrow c_1^2 \geq K^2$ (2)

(3) Kähler $\Rightarrow$ Hodge theory: hd Δ and Riem Δ are the same. The Hodge index theorem can be used to play (1) and (2) against each other and derive a contradiction, unless $c = \pm K$.

6) Theorem: [Lisca–Matić] $\xi_n^k \underset{\text{iso}}{\simeq} \xi_n^{k'} \Leftrightarrow k = k'$ or $k = n - k'$.

Proof: $W_n^k \subseteq S$ as in the lemma



Because isom. cont. structures have isom. collars, so we can glue both to S.

Gompf: every $\phi \in \mathrm{Diff}(\partial N_n)$ extends to N_n and acts as ± 1 on H^*.

This uses: (1) isotopy $\leftrightarrow$ homotopy for diffs (3D)
(2) do algebraic topology to classify diffeos.

So, id: $S \setminus W_n^k \to S' \setminus W_n^{k'}$ extends to a diffeo between S and S'. By diffeo invariance, $\phi^* \mathrm{Basic}(S') = \mathrm{Basic}(S)$

by fact above $\phi^*\{\pm c_1(S')\} = \{\pm c_1(S)\}$

So $\phi^* c_1(W_n^k) = \pm c_1(W_n^{k'}) = \pm(2k'-n)\,PD(T)$

by Gompf $= \pm c_1(W_n^k) = \pm(2k-n)\,PD(T)$

Other direction: by the form of the surgery: flip orientation on S.

So $k = k'$ or $n-k'$. $\square$

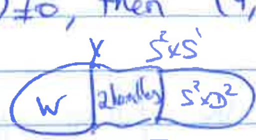
Steven: Fact above $\Rightarrow$ c_1(Kähler str) is an ~~invariant up to~~ diffeomorphism invariant of the manifold.

Discussion session: open problems

Q: classify (up to diffeo or symplectic deformation) fillings of any contact manifold not supported by a planar open book (or T^3).
We've seen McDuff and Wendl's theorems; they used some kind of foliation; the index was $n(2-2g)+2c_1(A)$ $\rightsquigarrow$ when genus bigger, the index drops, so it's hard to find foliations.

Theorem: If (Y,ξ) has a filling (W,ω) with $b_+^2(W)>0$ or $b_0^2(W)>0$ or $c_1(W,J)\neq 0$, then (Y,ξ) is not planar.
manifolds with boundary can have indefinite part.

Idea:



; find spheres in $S^2\times D^2$.
By rational ruled theorem, this is a blow-up of $S^2\times S^2$, so $W \hookrightarrow \#_n\overline{\mathbb{CP}^2}$, so need $b_+^2>0$. "B"

Rem: ind $= n(2-2g)+2c_1(A) = (n+2)(2-2g)+2A\cdot A$ by adjunction. If we want foliation, $A\cdot A=0$, so the index is negative for g larger than 1.

For many many examples of higher genus OBD, there are infinitely many minimal Stein fillings; could have (Yasi)
- ∞'ly many distinguished by H_*
- ∞'ly many Stein fillings all homeo but not diffeo: [Etnyre - Akhmedov - Smith - Park], [Baykur] by LF, [van Horn Morris]

Rem: there is no homeo class of 4-mflds where we have a classification of the smooth type.

Q: when does (Y,ξ) admit only finitely many fillings?
→ Planar case: are there always ∞'ly many factorizations of monodromy? We know it's true for lens spaces, Seifert fibered spaces, ... But that's pretty much it.
→ Genus 1: have certain types of bounds for fillings $\rightsquigarrow$ hope.
→ Genus ≥ 2: gets out of control. But, see next question.

Q: given (Y,ξ), what is the minimal genus of a supporting OBD?
Can always stabilize to raise genus, but there should be a minimal one.
By Wendl, some things have minimal genus >0 (fillings of planar are neg def and $c_1=0$)
Open question: does there exist (Y,ξ) with min. genus ≥ 2?
Canonical set of examples that people think might work: page = genus g with 1 ∂ component, monodromy = Dehn twist along boundary.

Rem: no formal obstruction to this: OT mflds admit planar OBD.

Q: is there a homology 3-sphere which is Stein fillable and supported by a planar OBD (other than S^3)?

To "destabilize": check out [Wand]'s stuff. He developed some technology to detect tightness from monodromy. He proved that Legendrian surgery preserves tightness.

Q: can you restrict the topology of the fillings of some contact 3-mfld in any way?
If (Y,ξ) has a contractible Stein filling, does this rule out some possibilities for (Y,ξ)?
Conjecture (Gompf): no Brieskorn sphere has a contractible Stein filling.
Possible approach: find a nice cap, and use some 4-mfld theorem (Donaldson's diagonalization) or symplectic theorem (McDuff's rational ruling)

Q: if (X^4,ω) has $c_1(TX,J)=0$ and $H_*(X;\mathbb{Z})=H_*(K3;\mathbb{Z})$, is it true that $X\cong K3$ diffeomorphism or symplectomorphism?
More generally, are there $(X_1,\omega_1)\not\cong(X_2,\omega_2)$ but $X_1 \underset{\text{diffeo}}{\cong} X_2$ sending $c_1(\omega_1)$ to $c_1(\omega_2)$? There are symp def equiv examples without the c_1 condition.

$\exists$ examples of Leg knots with same tb and rot not Leg isotopic; look at Stein mflds obtained from attaching handles to these. See [Etnyre-Honda]'s "Cable of torus knots" or [Tosun].

Q: $\exists$? Liouville cobordism structure on $[0,1]\times Y^3$ which is not $\lambda=e^r\alpha$. Wrong in higher dimensions (McLean): standard ball in exotic ball.
But in dim 4, there is no exotic structures on balls.

Ziva - Flexible Weinstein structures

Outline: 1) Intro to flexible Weinstein structures
2) Loose Legendrians
3) Classification of FWS
4) Examples

1) Weinstein manifold: $(W^{2n}, \omega, X, \phi)$: $n>2$
- W open manifold
- ω symplectic form
- X Liouville for ω, gradient-like for ϕ
- ϕ exhausting generalized Morse $W \to \mathbb{R}$

"Exhausting" = proper and bounded below ; "generalized" = critical points are non-degenerate or embryonic, "gradient-like" means that $X=0$ at $\mathrm{crit}(\phi)$ and $d\phi(X)>0$ away from $\mathrm{crit}(\phi)$.

Weinstein cobordism: W with $\partial W = \partial_- W \sqcup \partial_+ W$, such that $\partial_\pm W$ are regular level sets for ϕ. $\partial_+ W$
Weinstein domain: if $\partial_- W = \emptyset$.

Notes: * X Liouville $\Rightarrow \omega = d\lambda \Rightarrow \forall c \in \mathbb{R}$ regular for ϕ, $\alpha_c := \lambda|_{\phi^{-1}(c)}$ is contact on $\phi^{-1}(c)$.
* $\forall p \in \mathrm{crit}(\phi)$, W_p^- is isotropic (symplectically), and $W_p^- \cap \phi^{-1}(c)$ for c regular is isotropic (contactically)
$\Rightarrow$ for $p \in \mathrm{crit}(\phi)$, $\mathrm{ind}(p) \leq n$.

Flexibility: - cut W along regular level sets into elementary W. cobordisms, ie in each of them, there is no trajectory between critical points.

Definition: an elementary Weinstein cobordism is flexible if the attaching spheres of all index n handles form a loose Legendrian link in $\partial_- W$. The whole cobordism is flexible if it can be decomposed into flexible elementary cobordisms.

Rem: * subcritical $\Rightarrow$ flexible
* for $n=2$, extend by defining: W subcritical or $\xi_{\partial_- W}$ is overtwisted.

2) Loose Legendrians:
- local model
- satisfy an h-principle.