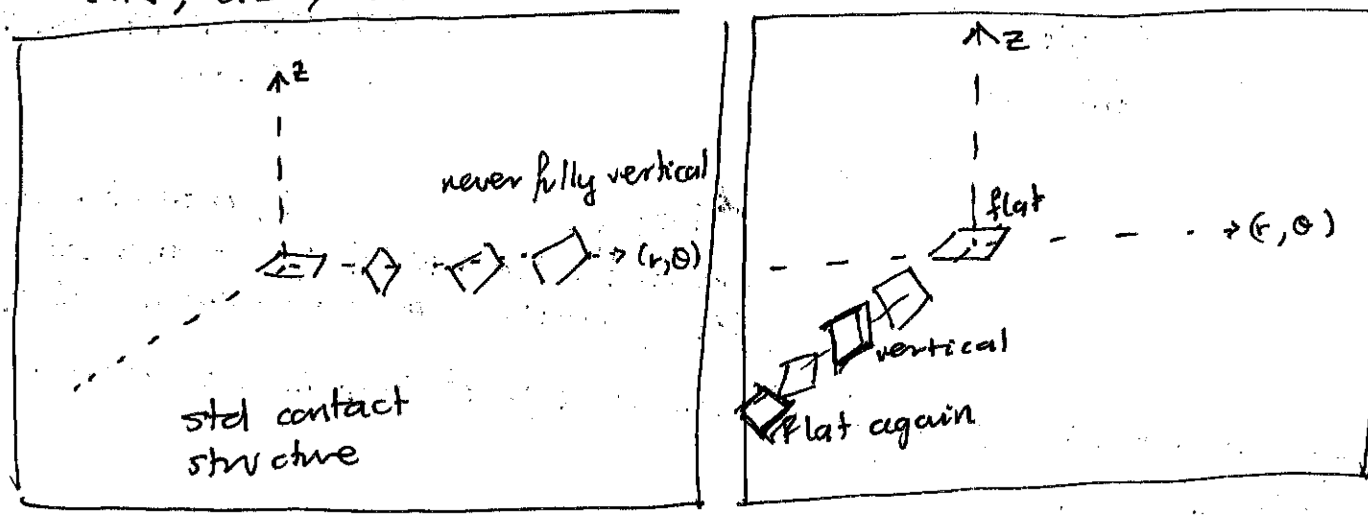
INTRODUCTION

Symplectic Fillings

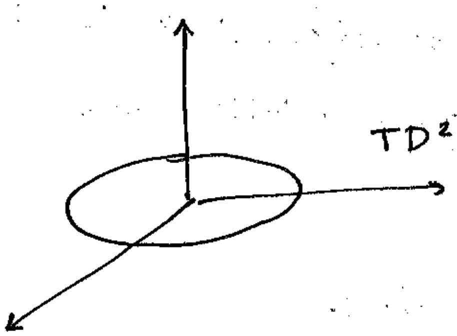
From contact viewpoint: 1982 Bennequin proves that $(\mathbb{R}^3, dz - y\,dx)$ is not contactomorphic to $(\mathbb{R}^3, \cos r\,dz + r\sin r\,d\theta)$



Newer viewpoint: (Gromov 85) shows that the compactifications (S^3, ξ_0) and (S^3, ξ_{OT}) are different because ξ_0 bounds and ξ_{OT} doesn't.

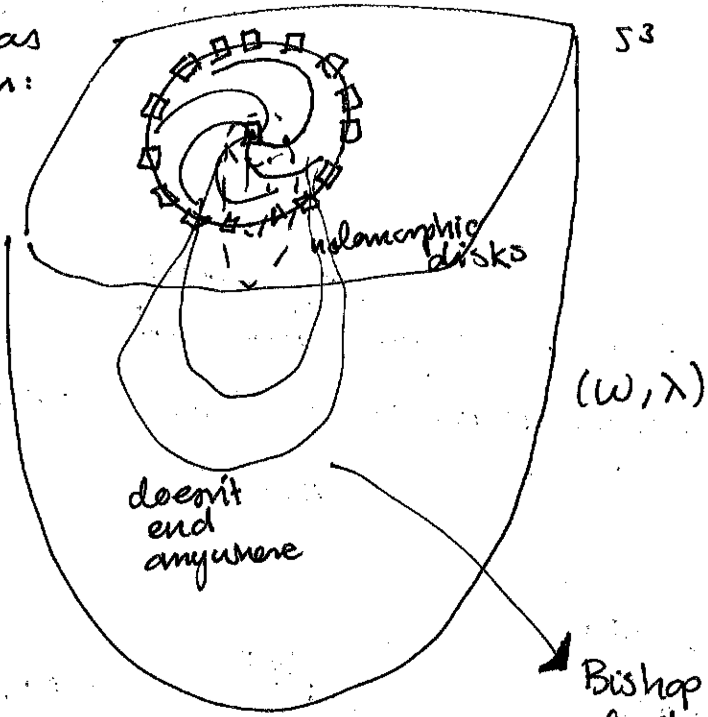
SKETCH OF PROOF:

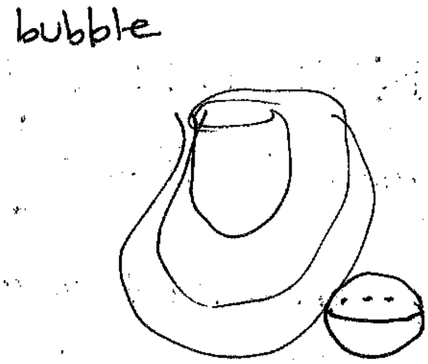
- (S^3, ξ_0) is bounds Darbax ball (D^4, ω_α)
- Suppose that (S^3, ξ_{OT}) bounds some symplectic mfld (which we can assume is nice e.g. (W, λ) where $d\lambda = \omega$).



$TD^2 \cap \xi_{OT}$ gives a line field w/ singularities where they coincide. (here is boundary + origin)

• Disk has foliation:

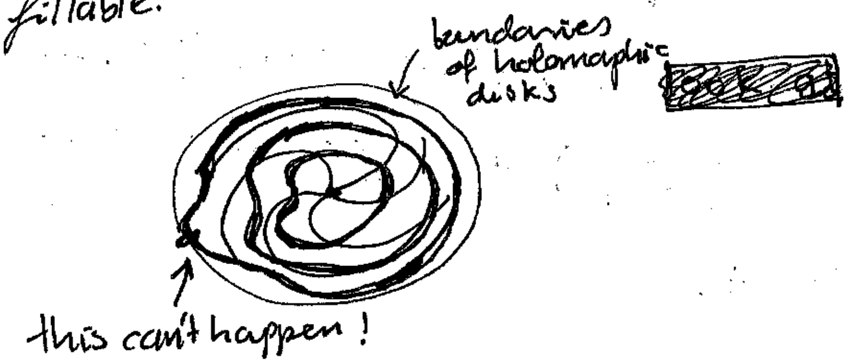




DOESNT EXIST BY EXACTNESS of (ω, λ)

Bishop 1-parameter fam
Explicit local model of holomorphic disks, which can never touch ∂D^2.

Remk: If you get rid of exactness still find that ξ_{OT} is not weakly fillable.



Remark: Not every $(2n+1)$ manifold has a smooth filling

E.g $\frac{SU_3}{SO(3)}$, $\mathbb{C}P^2 \times_{conj} S^1$

But $\Omega^{u}_{2n+1} = 0$, every manifold Y^{2n+1} which is almost contact is the boundary of a manifold W^{2n+2} with W almost complex.

Ω^{U}_{2n+1} = unitary bordism ring.

$\boxed{\text{Symplectic viewpoint}}$ Distinguish W_1 & W_2 via their boundaries which have same contact structure.

Consider the singular complex mfld of $\dim_{\mathbb{R}} = 6$

$$V = \{ z_0^7 + z_1^2 + z_2^2 + z_3^2 = 0 \} \subset \mathbb{C}^4$$

$\Rightarrow$ look at $\mathbb{C}^4 \setminus V$ & modify its topology

$\Rightarrow$ [McLean 08] $\widetilde{\mathbb{C}^4 \setminus V} \underset{\text{DIFF}}{\cong} \mathbb{R}^8$

Q: Is it symplectomorphic to $(\mathbb{R}^8, \omega_0)$?

Then, by looking at the boundaries (SH^+) you distinguish exact symplectic type; but this gives ω-invariant.

Q1: What about contact structures with NO symplectic fillings? Yes they exist: SW are a good obstruction for 3-mflds

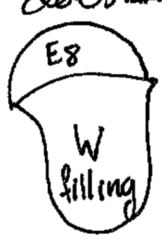
Glimpse:

$$\Sigma(2,3,5) = \{ z^2 + x^3 + y^5 = 0 \} \cap S^5 \subseteq \mathbb{C}^3$$

is the Poincare homology sphere

In fact $\partial E_8 = \Sigma_{2,3,5}$ & this one does not have a symplectic filling.

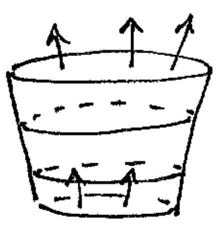
But $\Sigma(2,3,5)$ does.



$\Sigma(2,3,5)$ positive scalar curvature $\Rightarrow b_2^+(W) = 0$

smooth topology tells you that such a 4-mfld cannot exist $\square$

$Y \times I$



CAPS VS FILLINGS

$\uparrow$ ω expanding

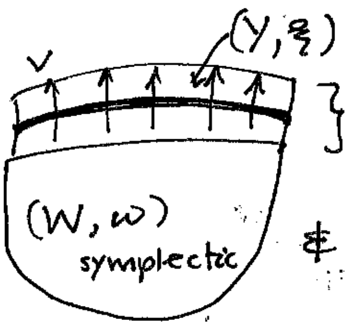
Caps 

(S^3, ξ_{OT}) has a cap. but it doesn't have a filling.

★ TYPES OF FILLINGS ★

~~STEIN~~

STRONG



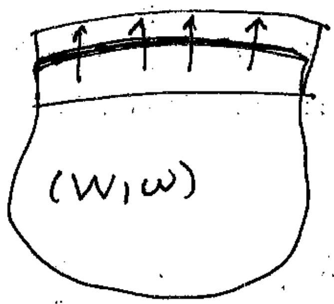
$\}\ \omega = d\lambda$

& exact near boundary

v must be upper transverse

~~Exercise show~~

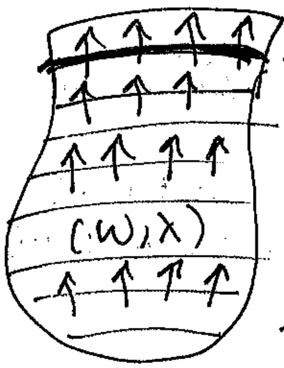
EXACT



$d\lambda = \omega$ on all of W

STEIN an exact filling that is well behaved wrt Morse function

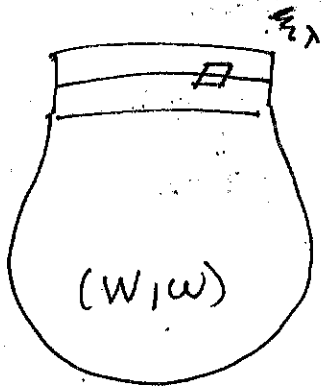
(Weinstein)



$\xrightarrow{\phi} \mathbb{R}$

level sets = contact mflds

{weak} (3d)



here you choose contact structure & then compare to symplectic structure.

No primitive, nor induced ξ.

does not allow for concatenation.

want
$\omega_x(\xi_x) > 0$

~~[illegible]~~ STEIN ⊆ EXACT ⊆ STRONG ⊆ WEAK

Q2: Are there contact manifolds with infinitely many fillings?

e.g (stein). $\boxed{\text{Yes}}$ we will have (Y^3, ξ) admitting a family of stein fillings X_n^4, with different homology groups. $H_0(X_n) = \mathbb{Z} \oplus \mathbb{Z}_m$.

- Sketch: try to factor elements of the mapping class group $\Gamma_{g,n}^k$ of a surface into two positive different ways.

Day 1: Lefschetz fibration + open book.

The inclusions are all strict

EXAMPLES in 3 dim:

(T^3, ξ_n)

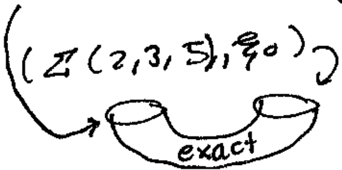
$\alpha_n = \cos(nx)dy + \sin(nx)dz$

~~[illegible]~~ stein ⊆ exact ⊆ strong ⊆ weak

∃ ~~[illegible]~~ contact structures

$(\Sigma(2,3,5), \xi_1)$ ③

$(\Sigma(2,3,5), \xi_0)$

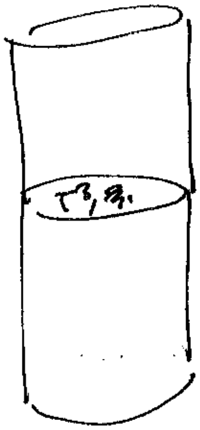


② Start w/ weak filling of a homology sphere ⇒ its a strong filling ~~but not~~ by cohomological conditions.

① THM 1: ξ_n are weakly fillable (by $T^2 \times D^2$)

(2) Only $\xi_1 = \partial(T^* T^2)$. is strongly fillable

(1) continued



$\Leftarrow$ somenthing $= \mathbb{R}^4$ at ∞.

n fold over $\rightarrow$

Take lagrangian T^2

T^2_{clifford}

$$\mathcal{O}_p(T^2_{\text{clifford}}) \subseteq (\mathbb{R}^4, \omega_0)$$
$$\parallel$$
$$T^*T^2$$

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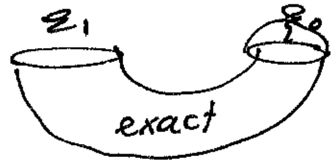
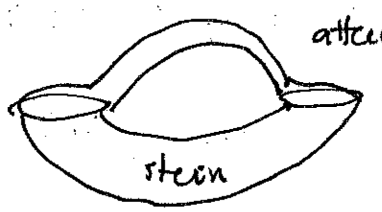
(3) continued



$\leftarrow$ can be capped

why isn't it stein?



attached a handle

Wanted to use:

THM: Given a stein filling of $(Y_1^3, \xi_1) \# (Y_2^3, \xi_2)$

$\Rightarrow (Y_i^3, \xi_i)$ is stein fillable.
