Scott - Contact manifolds with flexible fillings

Ref: [Lazarev]

Let (Y^{2n-1}, ξ) contact, W^{2n} Weinstein filling ⇝ have handles $\leq n$.

Theorem (Yau) if W_1, W_2 are 2 subcritical fillings, then $H^*(W_1) = H^*(W_2)$

Theorem 1 [Lazarev] if W_1, W_2 are flexible fillings of (Y^{2n-1}, ξ), then $H^*(W_1; \mathbb{Z}) \cong H^*(W_2; \mathbb{Z})$.

Rem: if Y has 1 flexible filling, then replacing W_2 by any Liouville filling with $SH = 0$ gives the same conclusion.

Theorem 2, if $n \geq 3$ and (Y^{2n-1}, ξ) has a flexible filling, then $\exists$ infinitely many (ξ_M) such that $\xi_M \not\cong \xi$, such that (Y^{2n-1}, ξ_M) has flexible fillings.

Proof: of theorem 1 ⇒ theorem 2: (Y, ξ) ∩ W, (S^{2n-1}, ξ) ∩ $\exists M$ as flexible fillings

Take a boundary connect sum (ie attach 1-handle):

⇝ $(Y, \xi) \# (S^{2n-1}, \xi) \simeq (Y, \xi)$

∩ as f.f.

filling of $W \natural M$ (flexible)

And $H^*(W \natural M) = H^*(W) \oplus H^*(M)$ so we can make their homology different. How to get these M?

→ n odd: take M = Brieskorn mfld, $\dim H^n(M_i) \to \infty$.

→ n even: [Geiges] "Applications of contact surgery" to find filling of spheres with $\dim H^n(M) \geq 1$ ⇝ take $M_i = \natural_i M$. □

Rem: $(Y, \xi) \overset{\text{homotopic}}{\sim} (Y, \xi_M)$.

Now, let' walk towards proving theorem 1.

Definition: (Y^{2n-1}, ξ) is dynamically convex (DC) if $\exists \alpha$ such that all Reeb orbits have positive degree: $|\gamma| = \mu_{CZ}(\gamma) + n - 3 > 0$.

Theorem: if DC, then SH^+ is indep. of the Stein filling, if $c_1 = 0$.

contact forms

Definition: (Y^{2n-1}, ξ) is asymptotically DC (ADC) if $\exists\ \alpha_1 \geq \alpha_2 \geq \dots$ and $D_1 < D_2 < \dots \to \infty$ such that $P^{<D_i}(Y, \alpha_i)$ (set of Reeb orbits of action $<D_i$) all have positive degree.

Theorem. if ADC, then SH^+ is indep. of the Stein filling, if $c_1 = 0$.

So: we want to show that flexible fillings are ADC:
$$\begin{tikzcd} H \arrow[r] & SH = 0 \arrow[dl] \\ SH^+ \arrow[u] & \end{tikzcd}$$
So $SH^+ \simeq H$, and SH^+ does not depend on W_1 or W_2, and hence $H^*(W_1) \simeq H^*(W_2)$.
Rem: $\alpha_1 \geq \alpha_2 \geq \dots$ is important; without that, being ADC would be trivial. ~~Since~~ This is a continuation map situation; if $\alpha_1 \geq \alpha_2$, can connect them by a piece of symplectization.

If (Y^{2n-1}, ξ) has flexible filling W^{2n}:

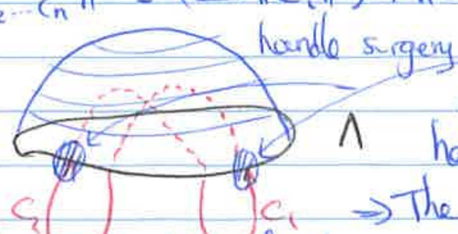
Theorem: if $(Y_-, \xi) \to (Y_+, \xi)$ through subcritical (Yau) or flexible (Oleg) surgery, then Y_- ADC $\Rightarrow$ Y_+ ADC.

Since (S^{2n-1}, ξ_{std}) is ADC, this would prove that being flexible is ADC, hence SH^+ indep of filling by theorem above. So, we just have to prove the theorem.
Rem: when you do surgery, that ~~deg~~ decreases the contact form. Being ADC is important; it wouldn't work for just DC.

Proposition: [Bourgeois-Ekholm-Eliashberg] after surgery along an attaching Legendrian sphere Λ^{n-1} $(n \geq 3)$, we have

* $\left\{ \begin{array}{l} \text{new Reeb orbits} \\ \text{with period} < D \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{cyclic words of Reeb chords} \\ \text{in } Y \text{ on } \Lambda \text{ with period} < D \end{array} \right\}$
* $\|\gamma_{c_1 c_2 \cdots c_n}\| = \left(\sum \|c_i\|\right) + n - 3$

"Proof":



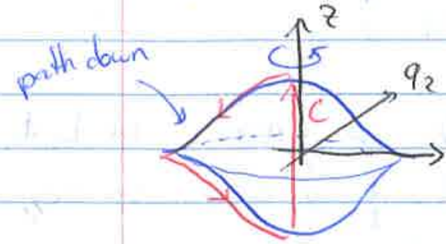
Given c_1, c_2: perturb so that end of c_1 flows to beginning of c_2 in the handle, and end of c_2 to beginning of c_1.
$\Rightarrow$ The resulting orbit is close to $\underbrace{c_1 \text{ and } c_2}_{}$.
c_1 and c_2

Conversely, an orbit clearly gives a word. □

Rem: the Reeb flow looks like the geodesic flow on the disk. In particular, all orbits leave the disk.

Key lemma: if $\Lambda^{n-1} \subseteq Y$ $(n \geq 3)$ is loose, $\exists$ Legendrian isotopy such that (period-bounded) Reeb chords have positive degree.

In $(\mathbb{R}^{2n-1}, dz - \sum p_i dq_i)$, ex $S^2 \subseteq \mathbb{R}^5$, consider



R_α is $\partial/\partial z$, so we want vertical lines in the front proj. between points with the same slope.

Given c a Reeb chord, follow it, and then choose a path that comes back down:

$$|c| = \underset{\#\text{down cusps}}{D} - \underset{\#\text{up cusps}}{U} + \operatorname{ind}_{h_2 - h_1}(p) - 1,$$ where h_1 and h_2 are the functions giving z in terms of q_i's.

$$= 1 - 0 + 2 - 1 = 2$$

To increase number of downward cusps: stabilize.

But we introduce new Reeb chords (red). For them:

$$|c| = 2 - 0 + (\operatorname{ind} \geq 0) - 1 \geq 1,$$ so it's ok.

Also, since Λ is loose, one can realize this stabilization by an isotopy.

Rem: we have this decreasing sequence of contact forms because of this stabilization, and because ~~attach~~ doing surgery also tends to make the handles thinner.