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COMPUTATIONS ON BRIESKORN MANIFOLDS.

Goal: Use SH^+ to distinguish contact structures on Brieksorn manifolds.

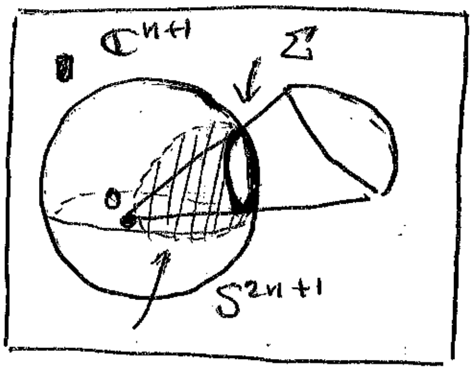
(1) Brieskorn mflds

(2) Ustilovsky's exotic spheres

Interlude Morse-Bott things

(3) Uebele's computation of SH^+(filling of $\Sigma(2l,2,\dots,2)$)

DEFN: $\Sigma(a_0,\dots,a_n) = \{ z_0^{a_0} + \dots + z_n^{a_n} = 0 \} \cap S^{2n+1} \subseteq \mathbb{C}^{n+1}$



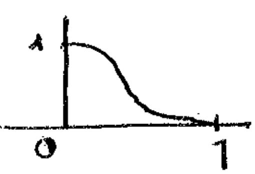
$$\alpha_a = \frac{i}{8} \sum a_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

$$R_{\alpha_a} = \left(\frac{4i}{a_0} z_0, \dots, \frac{4i}{a_n} z_n \right)$$

$$\phi_t^a(z) = \left(e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n \right)$$

Filling $\{ z_0^{a_0} + \dots + z_n^{a_n} = \varepsilon \phi(\|z\|) \} \cap D^{2n+2} \subseteq \mathbb{C}^n$

ϕ



For ε small this is an exact filling which is parallelizable $c_1(\omega_a) = 0$

THM (1) $\pi_1(\Sigma(a)) = \dots = \pi_{n-2}(\Sigma(a)) = 0$

(2) $\exists$ conditions that on a $\Leftrightarrow \Sigma(a) \underset{\text{homeo}}{\simeq} S^{2n-1}$

[Randell] algorithm to compute $H_{n-1}(\Sigma(a))$

Fun facts

1) $\Sigma(2,2,2,3,6k-1)$ for $k=1,\dots,28$ are all smooth S^7's.

2) Any M^5 spin with $\pi_1(M)=0$ is a connected sum of $\Sigma(a^{(m)})$'s.

[redacted] THM [Brieskorn] If $p \equiv \pm 1 \mod 8$, $\Sigma(p, \underbrace{2,\dots,2}_{2m+1})$ is diffeo to S^{4m+1}_{std} and it has α_p

THM [U?] $p_1 \neq p_2 \Rightarrow \xi_{p_1} \not\cong \xi_{p_2}$

"Proof" 1) Find an explicit [redacted] perturbation of α_p.

2) Computes the degrees of Reeb orbits in $CC_*(\Sigma(p), \xi_p) \Rightarrow$ even

3) $\partial = 0 \Rightarrow CH_*$

DEFN: an almost contact structure on Y^{2n+1} is (α, β) in $\Omega^1(Y) \times \Omega^2(Y)$ s.t. β is nondegenerate on $\ker\alpha$. $\Leftrightarrow$ reducing str(TY) to $U(n)\times 1$.

DEFN: we say (S^{2n+1}, ξ) exotic if $\underset{\text{contactomorphic}}{\not\cong} (S^{2n+1}, \xi_{std})$ and (S^{2n+1}, ξ) trivial if homotopic to (S^{2n+1}, ξ_{std}) through an almost complex structure.

$$\begin{tikzcd} & B(\mathcal{U}(2n)\times 1) \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & BSO(4n+1) \end{tikzcd}$$

$$\pi_{4m+1}\left(SO(4n+1) / U(2n)\times 1 \right) =: G = \left\{ \begin{array}{c} \text{almost contact} \\ \text{structures} \end{array} \right\}$$

[Massey] showed that G is cyclic of order $d = \begin{cases} (2m)! & m \text{ even} \\ (2m)!/2 & m \text{ odd} \end{cases}$

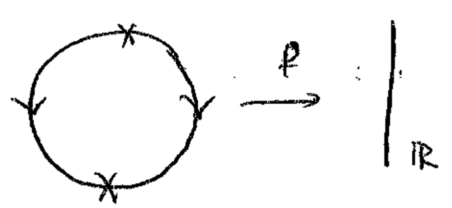
[Morita] $[\xi_p]$ is $\frac{p-1}{2}$ mod d, ξ_p on $\Sigma(p,2,\dots,2)$.

$\Rightarrow$ If $p = 1$ mod $2(2n)!$ and $p \equiv \pm 1$ mod 8 then

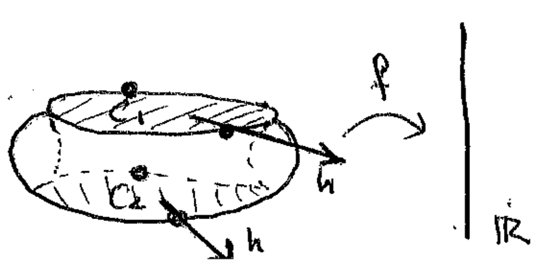
ξ_p is homotopically std.

THM $\exists$ infinitely many exotic homotopically [redacted] trivial contact structures on S^{4n+1}

MORSE-BOTT: HM: generators = critical points
differential = graded flow lines



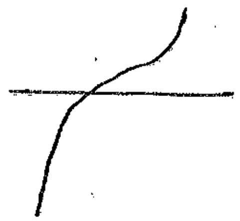
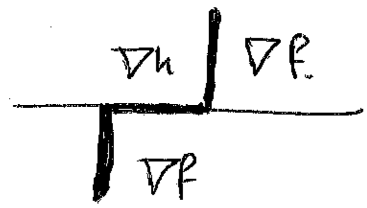
DEFN: We say that a function $f: M \to \mathbb{R}$ is Morse Bott if $\operatorname{crit}(f) = \bigsqcup C_i$ where C_i are submflds of M s.t. $\operatorname{Hess}_p f|_{\nu C_i}$ is nondegenerate for $p \in C_i$.



Pick h Morse function on $\mathrm{crit}(f)$

then $f+\varepsilon\rho h$ where ρ is a cutoff function near $\mathrm{crit}(f)$.

$\mathrm{crit}(f+\varepsilon\rho h) = \mathrm{crit}(h)$

c_i  , let $\varepsilon \mapsto 0$ 

SH^+_{MB}: (Σ, ξ) contact w/ filling W; let H be a quadratic Hamiltonian, C^2-small on W

[Bourgeois, Oancea] SH^+_{MB} when S^1-families

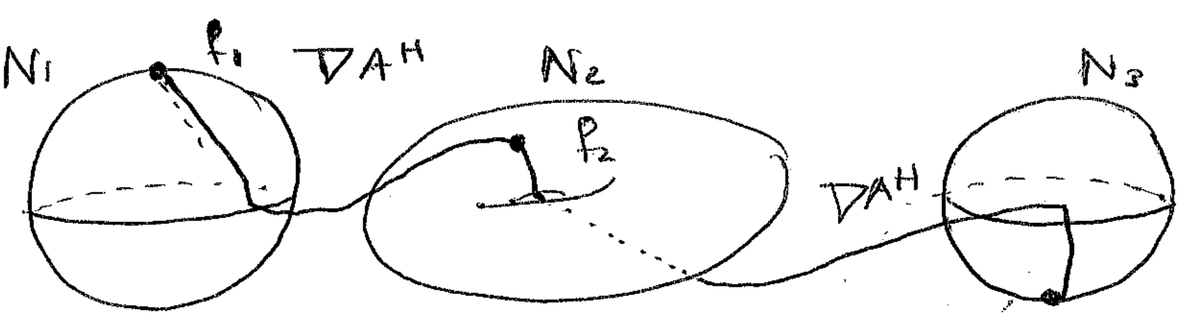
MB condition: $N_T = \{z \in \Sigma \mid \phi_T(z) = z\}$ closed submfld, such that $rk(d\alpha|_{N_T})$ constant and

$T_p N_T = \ker(D\phi_T - id)$

GRADING: * $c_1(W) = 0$

* all Reeb orbits are contractible.

Gen: (T, η) $\eta \in \mathrm{Crit}(f_T)$, f_T Morse for N_T

Diff 

<u>Grading</u>: $|(\tau,\eta)| = \mu_{cz}(N_T) + \mathrm{ind}_\ell(\eta) - \frac{1}{2}(\dim N_T - 1)$

(3) Uebeles thesis: $\Sigma_\ell^n = \Sigma(2\ell, \underbrace{2, \dots, 2}_{n \text{ odd}})$

$$\alpha_\ell^n = \frac{i\ell}{4}(z_0 d\bar{z}_0 - \bar{z}_0 dz_0) + \frac{i}{4}\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j)$$

$$R_{\alpha_\ell^n} = 2i\left(\frac{z_0}{\ell}, z_1, \dots, z_n\right)$$

$$\phi_t(z) = \left(e^{2i/\ell} z_0, \dots, e^{2it} z_1, \dots, e^{2it} z_n\right)$$

Randall's: $H_{n-1}(\Sigma_\ell^n) \cong \mathbb{Z}$

Walls: Σ_ℓ^n diffeo $\cong \begin{cases} S^{n-1} \times S^n & \text{if } \ell \equiv 0 \mod 3 \\ S^* S^n & \text{if } \ell \equiv 1 \mod 3 \\ (S^{n-1} \times S^n) \# K & \ell \equiv 2 \mod 3 \\ S^* S^n \# K & \ell \equiv 3 \mod 3 \end{cases}$

If $n = 3$: $S^2 \times S^3$

$\boxed{\text{If } n > 3}$ want to show $\Sigma_n^\ell \not\cong \Sigma_n^{\ell'}$ $\ell = \ell'$

LEMMA for Σ_ℓ, SH^+ is independent of filling as long as $c_1 = 0$

1) ϕ_t periodic

If $z_0 = 0$ orbit has period π

$z_0 \neq 0$ orbit has period $\ell\pi$

$$\Rightarrow N_T = \begin{cases} \Sigma_n^\ell & \text{if } T = N\pi \ \& \ \ell \mid N \\ \Sigma_n^\ell \cap \{z_0 = 0\} & \text{if } T = N\pi \ \& \ \ell \nmid N \\ \emptyset & \text{else.} \end{cases}$$

2) Morse functions on N_T: $\Sigma_n^\ell \cap \{z_0 = 0\} = S^* S^{n-1}$

$\rightsquigarrow$ $\exists$ perfect Morse function on $S^* S^{n-1}$ &

index of critical points $= \{0, n-1, n-2, 2n-3\}$

* Σ_n^ℓ: pretend there is perfect Morse function

ind $\in \{0, n-1, n, 2n-1\}$

$$|(\tau, n)| = \mu_{cz}(N_T) + \mathrm{ind}_{f_T}(\eta) - \tfrac{1}{2}(\dim N_T - 1)$$

3) For γ an orbit of length $N\pi$, we find that

$$\mu_{cz}(\gamma) = \begin{cases} 2\frac{N}{\ell} + 2N(n-2) & \text{if } \ell \mid N \\ 2\lfloor \frac{N}{\ell} \rfloor + 2N(n-2) + 1 & \text{if } \ell \nmid N \end{cases}$$

How: trivialize $\xi_\ell^\perp$

NOW: DEGREES OF GENERATORS; COMPUTE.

(5) Compute ∂: From a critical submfld N_T to N_T $\partial = 0$

from N_T to $N_{T'}$: index(of generator w/ period $N+1$) $\geq$ index(generator w/ period N) $- 1$

$(n \geq 3)$ $\Downarrow$ $\partial = 0$