Cédric - Computations on Brieskorn manifolds

Goal: use SH^+ to distinguish contact structures

Plan: 1) Brieskorn manifolds

2) Ustilovsky's exotic spheres

♪ Interlude ♪ Morse-Bott things

3) Uebele's computation of SH^+ for fillings of $\Sigma(2l, 2, \dots, 2)$.

1) Brieskorn manifolds.

Definition. $\Sigma^{2n-1}(a_0, \dots, a_n) := \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0\} \cap S^{2n+1} \subseteq \mathbb{C}^{n+1}$.

Write $a = (a_0, \dots, a_n)$ and $\Sigma(a)$ for $\Sigma(a_0, \dots, a_n)$

$\rightsquigarrow$ Contact form on $\Sigma(a)$: $\alpha_a = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$.

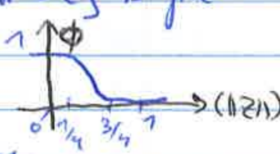
Rem: $\mathbb{C}^{n+1}$ admits an a-weighted Hermitian form: $\langle \xi, \zeta \rangle_a = \frac{1}{2} \sum_{k=0} a_k \xi_k \bar{\zeta}_k$, with associated (wrt i) symplectic form $\omega_a = -\mathrm{Im}\langle \cdot, \cdot \rangle_a = \frac{i}{4} \sum a_k dz_k \wedge d\bar{z}_k$.

Then $\frac{z}{2}$ is Liouville for ω_a, with Liouville 1-form $\lambda_a = \omega_a(\frac{z}{2}, -)$ $= \frac{i}{8} \sum a_k (z_k d\bar{z}_k - \bar{z}_k dz_k)$.

$\rightsquigarrow$ Reeb vector field: $R_{\alpha_a} = \left(\frac{4i}{a_0} z_0, \dots, \frac{4i}{a_n} z_n\right)$

$\rightsquigarrow$ Reeb flow: $\phi_t^a(z) = (e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n)$

$\rightsquigarrow$ Filling: we'd like to fill $\Sigma(a)$ by $\{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0\}$, but this has a singularity at 0. However, $\{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = \varepsilon\}$ is fine. We take an interpolation: $W_a := \{z_0^{a_0} + \dots + z_n^{a_n} = \varepsilon \phi(\|z\|)\}$.



For ε small, this is an exact filling that is parallelizable, hence $c_1(W_a) = 0$. Also, it has the homotopy type of $\bigvee_{\mu(a)} S^n$ (wedge product), where $\mu(a) = \prod_{i=0}^{n} (a_i - 1)$.

Theorem: 1) $\pi_1(\Sigma(a)) = \dots = \pi_{n-2}(\Sigma(a)) = 0$, and there is an algorithm ([Randell]) to compute $H_{n-1}(\Sigma(a))$.

$n \geq 3$ 2) $\Sigma(a)$ is homeomorphic to $S^{2n-1} \iff \exists\, a_i, a_j$ which are relatively prime to all other exponents, OR $\exists\, a_i$ which is relatively prime to all the other exponents, and a set $\{a_{j_1}, \dots, a_{j_r}\}$ ($r \geq 3$ odd) such that each a_{j_k} is relatively prime to any exponent not in the set, while $\gcd(a_{j_k}, a_{j_l}) = 2$ $\forall k \neq l$.

Fun facts: 1) $\Sigma(2, 2, 2, 3, 6k-1)$ for $k = 1, \dots, 28$ give all 28 smooth structures on the oriented S^7.

2) Any M^5 spin with $\pi_1(M) = 0$ is a connect sum of $\Sigma(a)$'s.

2) Ustilovsky's exotic spheres

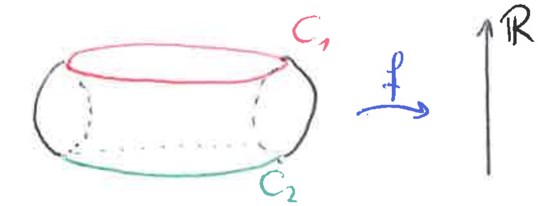
Let $(\Sigma^m_p, \xi^m_p) := (\Sigma(p, \overbrace{2,\dots,2}^{2m+1}), \ker \alpha_{(p,2,\dots,2)})$, with an odd number of 2's.
By the criterion above, these manifolds are homeomorphic to spheres.
[Brieskorn]: for $p \equiv \pm 1 \bmod 8$, Σ^m_p is diffeomorphic to S^{4m+1} standard.
Recall: $\alpha^m_p = \frac{ip}{8}(z_0 d\bar{z}_0 - \bar{z}_0 dz_0) + \frac{i}{4}\sum_{j=1}^{2m+1}(z_j d\bar{z}_j - \bar{z}_j dz_j)$, $\xi^m_p := \ker(\alpha^m_p)$

Theorem (Ustilovsky) $p_1 \neq p_2 \Rightarrow \xi^m_{p_1}$ is not contactomorphic to $\xi^m_{p_2}$.
"Proof": use contact homology. Take an explicit perturbation of α^m_p so that all the Reeb orbits are non-degenerate. Compute their degrees; find that they are all even. Since the degree of the differential is -1, it vanishes, hence contact homology is isomorphic to the contact algebra. For different values of p, the degrees of the generators differ, hence contact homologies are not isomorphic. But contact homology is an invariant of the contact structure. $\square$

Now, we will see that infinitely many of those are "homotopically equal".
Definition: an almost contact structure on Y^{2n+1} is a pair (α, β) where α is a 1-form and β a 2-form, such that $\alpha \wedge \beta^n$ is non-vanishing. This is the same as a reduction of the structure group of TY to $U(n) \times 1$.
ex: a contact structure $\xi = \ker \alpha$ gives the almost contact structure $(\alpha, d\alpha)$.
The homotopy class among almost contact structures of a contact structure is called its formal class.
Definitions: a contact structure on S^{2n+1} is
* exotic if it is not contactomorphic to (S^{2n+1}, ξ_{std})
* homotopically trivial if it is in the formal class of (S^{2n+1}, ξ_{std})

An almost contact structure on S^{4m+1} $\longleftrightarrow$ lift
$$\begin{tikzcd} & B(U(2m)\times 1) \arrow[d] \\ S^{4m+1} \arrow[r] \arrow[ur, dashed] & BSO(4m+1) \end{tikzcd}$$
and those are classified by $G := \pi_{4m+1}(SO(4m+1)/U(2m)\times 1)$, since $SO(4m+1)/U(2m)\times 1$ is the fiber of the vertical map.
[Massey]: G is cyclic of order $d = \begin{cases} (2m)! & \text{if } m \text{ is even} \\ \frac{(2m)!}{2} & \text{if } m \text{ is odd} \end{cases}$
[Morita]: $[\xi^m_p]$ above is $\frac{p-1}{2} \bmod d$ in that group.
$\Rightarrow$ for $p \equiv 1 \bmod 2\cdot(2m)!$, ξ^m_p is homotopically standard. So we get:

Theorem: $\exists$ infinitely many homotopically trivial exotic contact structures on S^{4m+1}.
Proof: for fixed m, $\exists$ infinitely many p st $p \equiv \pm 1 \bmod 8$ and $p \equiv 1 \bmod 2\cdot(2m)!$ $\square$

ex: 

♪ Interlude ♪: Morse-Bott things

In finite dimension, how to do Morse homology if the critical points of $f: M \to \mathbb{R}$ are not isolated?

Definition: $f: M \to \mathbb{R}$ is Morse-Bott if $\text{crit}(f) = \bigsqcup C_i$ is a disjoint union of connected submanifolds, such that $\text{Hess}_p f|_{\nu(C_i)}$ is non-degenerate $\forall p \in C_i$. ex: see above.

What we could do to compute Morse-homology is choose h a Morse function on $\text{crit}(f)$; then $f + \varepsilon \rho h$ is Morse (for ε small and ρ an appropriately chose cutoff near $\text{crit}(f)$), and its critical points are exactly those of h. But we'd rather not break the symmetry, for the sake of computability.

Idea: crit f $\overset{\varepsilon \to 0}{\rightsquigarrow}$ crit f, so maybe counting these "cascades" could work...
grad flow of $f + \varepsilon\rho h$ — grad flow of f — grad flow of h

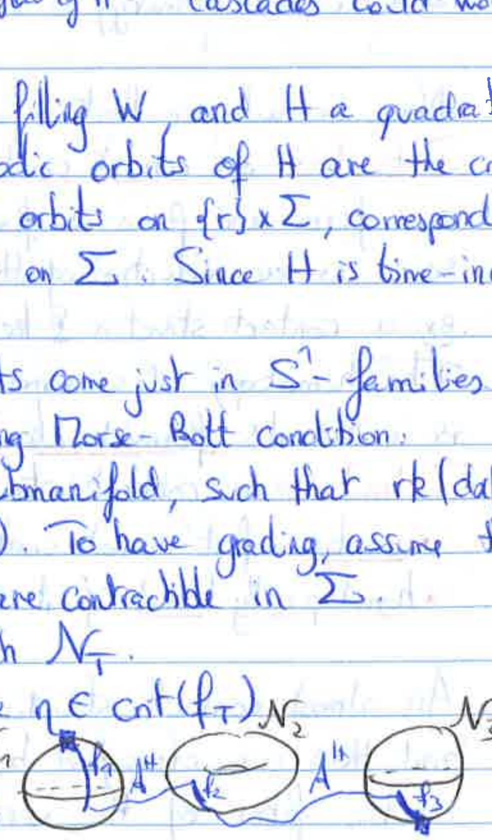
↳ For SH^+: let (Σ, ξ) contact with filling W and H a quadratic Hamiltonian, C^2-small in W. The periodic orbits of H are the critical points of H in W, and the 1-periodic orbits on $\{r\} \times \Sigma$, corresponding to the closed Reeb orbits of period $h'(e^r)$ on Σ. Since H is time-independent, these come at least in S^1-families.

[Bourgeois-Oancea]: $SH_{\text{Morse-Bott}}$ if orbits come just in S^1-families. But it also works here, under the following Morse-Bott condition: $N_T = \{z \in \Sigma \mid \phi_T(z) = z\}$ is a closed submanifold, such that $\text{rk}(d\alpha|_{N_T})$ is locally constant and $T_p N_T = \ker(d\phi_T - \text{id})$. To have grading, assume that $c_1(W)$ and that the closed Reeb orbits are contractible in Σ.

Choose a Morse function f_T on each N_T.

→ Generators of $SC^+(W)$: (T, η) where $\eta \in \text{crit}(f_T)$

→ Differential: Counts isolated trajectories of the form in the picture, where the lines "A" denote Floer cylinders



(*) → Grading: $\mu(T, \eta) = \mu_{CZ}(N_T) + \text{ind}_{f_T}(\eta) - \frac{1}{2}(\dim N_T - 1)$.

Why? 1) If we perturb $f + \varepsilon \rho h$, that is what we get.
2) With this grading, the differential has degree -1.

♪ End of interlude ♪

3) Uebele's computations:

We now focus on $\Sigma_\ell^n = \Sigma(2\ell, \overbrace{2, \dots, 2}^{n \text{ odd}})$. This has $R_\alpha = 2i\left(\frac{z_0}{\ell}, z_1, \dots, z_n\right)$ and $\phi_t(z) = \left(e^{2it/\ell} z_0, e^{2it} z_1, \dots, e^{2it} z_n\right)$, by the formulae in section 1).

$\to$ Randell's algorithm gives $H_{n-1}(\Sigma_\ell^n) \simeq \mathbb{Z}$.

$\to$ Wall's classification of highly-connected mflds: $\Sigma_\ell^n \underset{\text{diffeo}}{\cong} \begin{cases} S^{n-1} \times S^n & \text{if } \ell \equiv 0 \bmod 4 \\ S^*S^n & \ell \equiv 1 \bmod 4 \\ (S^{n-1} \times S^n) \# K & \ell \equiv 2 \bmod 4 \\ S^*S^n \# K & \ell \equiv 3 \bmod 4 \end{cases}$,

where $K = \Sigma(3, 2, \dots, 2)$ is the Kervaire sphere.

Rem: if $n = 3$, all give $S^2 \times S^3$, as $K \cong S^5_{std}$.

Theorem: the manifolds Σ_ℓ^n are pairwise non-contactomorphic.

To prove this, we compute SH^+(standard filling), and use the following lemma:

Lemma: for Σ_ℓ^n, the SH^+ is independent of the chosen filling W, as long as $c_1(W)|_{\pi_2(W)} = 0$.

So, let us compute SH^+, using the Morse-Bott setup presented above.

1) Critical manifolds: ϕ is periodic everywhere: if $z_0 = 0$ then period π, and if $z_0 \neq 0$, then period $\ell\pi$. So, we get: $N_T = \begin{cases} \Sigma_\ell^n & \text{if } T = N\pi, \ \ell | N \\ \Sigma_\ell^n \cap \{z_0 = 0\} & \text{" }, \ \ell \nmid N \\ \emptyset & \text{else.} \end{cases}$

2) Morse functions on N_T:

* $\Sigma_\ell^n \cap \{z_0 = 0\} \simeq S^*S^{n-1} \Rightarrow \exists$ perfect Morse fct, with indices $\in \{0, n-2, n-1, 2n-3\}$
* $\Sigma_\ell \Rightarrow$ [Fauck]: can pretend " " " " " " $\in \{0, n-1, n, 2n-1\}$

3) To compute Robbin-Salamon indices: by additivity of μ_{CZ}, we do it on $T\mathbb{C}^{n+1}$ and on $\xi_\ell^\perp$, which is generated by 4 explicit vector fields. For $N_{N\pi}$, get:

$\to$ On $T\mathbb{C}^{n+1}$: $\Phi^t = D\phi^t = \operatorname{diag}\left(e^{\frac{4it}{2\ell}}, \dots, e^{\frac{4it}{2}}\right) \rightsquigarrow \mu_{CZ}(\Phi) = \sum_{k=0}^{n} \left(\left\lfloor \frac{4 \cdot N\pi}{2\pi a_k} \right\rfloor + \left\lceil \frac{4N\pi}{2\pi a_k} \right\rceil \right)$

$\to$ On $\xi_\ell^\perp$: $\Phi_2^t = \operatorname{diag}(e^{4it}, 1)$ for some explicit basis of $\xi_\ell^\perp$.

$\Rightarrow \mu_{CZ}(\gamma \in N_{N\pi}) = \begin{cases} 2N/\ell + 2N(n-2) & \text{if } \ell | N \\ 2\lfloor N/\ell \rfloor + 2N(n-2) + 1 & \text{if } \ell \nmid N. \end{cases}$

4) Degrees of generators for SC^+: use the formula $(*)$ above

$\to \ell | N$: get $2\frac{N}{\ell} + 2N(n-2) + \{-n+1, 0, 1, n\}$

$\to \ell \nmid N$: get $2\lfloor \frac{N}{\ell} \rfloor + 2N(n-2) + \{-n+3, 1, 2, n\}$

5) The differential: within a critical manifold, the Morse function we chose is perfect, so the differential vanishes. From 1 mfld to the other: if $n \geq 5$, can see that $\exists!$ 1 pair of generators for each consecutive degrees, achieved by: ind(gen. with period $N+1$) = ind(gen. with period N) $- 1$. But ∂ decreases the period, hence $\partial = 0$. For $n = 3$, can use a symmetry to prove that a lot of the differentials still vanish.

$\Rightarrow$ So $SH^+ = SC^+$ (or close to it if $n = 3$), and by looking at the degrees, we see they differ for different values of ℓ. $\square$