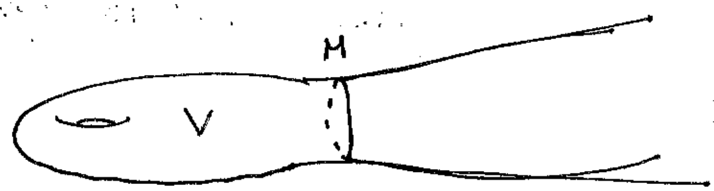
$\boxed{SH \ \& \ SH^*}$

<u>Goal</u>: THEOREM: If W, W' are two subcritical Stein fillings of (M, ξ) with $c_1(W) = c_1(W') = 0$

$$\Rightarrow H^*(W) \cong H^*(W').$$

SH & SH^* invariants of Liouville domain (V, λ)
submfld w/ primitive $d\lambda$ that is symplectic & a Liouville vector field Z, s.t. $i_Z d\lambda = \lambda$ & Z is positively transverse to ∂V. (Then $\alpha := \lambda|_{\partial V}$ is contact)

Let $\partial V = M \Rightarrow (\hat{V}, \hat{\lambda}) = (V, \lambda) \cup_{\partial} ([0,\infty) \times M, e^r \alpha)$



We can define the action functional

Pick $H : \hat{V} \to$

Define $\mathcal{A}^H : C^\infty(\mathbb{R}/\mathbb{Z}, \hat{V}) \longrightarrow \mathbb{R}$

$$\mathcal{A}^H(x) = \int_{S^1} x^* \hat{\lambda} - \int_{S^1} H(x(t))\,dt.$$

$\underset{\overset{\shortparallel}{S^1}}{x : \mathbb{R}/\mathbb{Z}} \to \hat{V}$ a loop.

$$d_x \mathcal{A}^H \cdot \xi = \int_{S^1} d\lambda(\xi(t), \underbrace{\dot{x}(t) - X^H(x(t))})\,dt$$

Critical points of $\mathcal{A}^H \longleftrightarrow x : S^1 \to \hat{V}$ $\quad \dot{x}(t) = X^H(x(t))$.

$J : T\hat{V} \to T\hat{V}$ $\quad J^2 = -1$ $\quad d\lambda(\cdot, J\cdot)$ is a Riemannian metric.

Now $\nabla_x \mathcal{A}^H = -J(\dot{x} - X^H(x))$

Now we can do Morse theory.

$\mathcal{U} : \mathbb{R}_s \to C^\infty(S^1, \hat{V})$

$\partial_s U = \nabla_x \mathcal{A}^H \circ U$.

~~Upward gradient flow~~

$u : \mathbb{R} \times S^1 \to \hat{V}$

$$\partial_s u + J(\partial_t U - X^H) = 0$$

Morse flow equ for this functional called Floer's equation.

Then, just do Morse theory

Form a complex

$$CF(H) = \bigoplus_{x \in \mathrm{Crit}(\mathcal{A}^H)} \mathbb{F}_2 \cdot x \quad ;$$

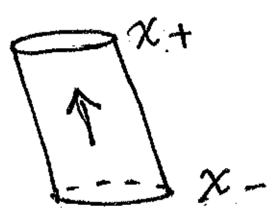
~~grading~~

$\boxed{c_1(V) = 0}$ $\quad CZ(x) \in \mathbb{Z}, \ |x| = CZ(x) - n$

$$d : CF_k(H) \to CF_{k-1}(H)$$

$$d(x_+) = \sum_{\substack{x \in \mathrm{Crit}(A_k) \\ |x_-| = |x_+| - 1}} \#_{\mathbb{F}_2} \mathcal{M}(x_-, x_+)\, x_-$$

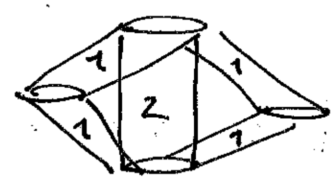
$$\to \left\{ U : \begin{array}{l} \partial_s U + J(\partial_t U - X^H) = 0 \\ \lim_{s \to \pm\infty} U(s,t) = x_\pm(t) \end{array} \right\} \Big/ \mathbb{R}$$



~~This doesn't work b/c we don't know of~~

index diff 1 : want finitely many

index diff 2 : degenerate into index diff 1



Compactness: Gromov Compactness works if the following hold

(a) uniform C^0-bounds

(b) uniform energy bounds. $\longrightarrow E(u) = \mathcal{A}^H(x_+) - \mathcal{A}^H(x_-)$

DEFN: The spectrum $\mathrm{Spec}(M, \alpha) = \{ T \in \mathbb{R} \mid \exists$ a Reeb orbit in α with period $T \}$

don't have to be simple.

DEFN: The space of admissible Hamiltonians

$$Ad(V, \lambda) = \cancel{\left\{ H : \hat{V} \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{outside a compact set} \\ Z(H) = H \text{ and} \\ \mathrm{slope}(H) \notin \mathrm{Spec}(M, \alpha) \end{array} \right\}}$$

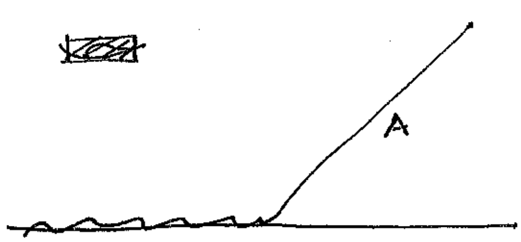
$Z = \partial_r$

$$= \left\{ H : \hat{V} \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{on the cylindrical end} \\ H(r, y) = A e^r + B \\ \mathrm{slope}(H) = A \notin \mathrm{Spec}(M, \alpha) \end{array} \right\}.$$

here cylindrical end

$[0, \infty) \times M$

$(r, \; y)$.



DEFN: A $J : T\hat{V} \to T\hat{V}$ is called cylindrical if on the cylindrical end

$$J = j \oplus J_\xi \; : \; j\partial r = R \quad J_\xi : \xi \to \xi \quad J_\xi^2 = -1$$

~~Ps~~ these choices ensure C^0-bounds ⇒ we have compactness ⇒ d is well defined & $d^2 = 0$.

thanks to the maximum principle. Not easy.

Now we can define $HF_k(H)$.

DEFN $HF_k(H) = H_k(CF_k(H), d)$.

$\Omega \subseteq \mathbb{R} \times S^1$, let $h(t) = At + B$

$$u(s,t) = (a(s,t), v(s,t)).$$

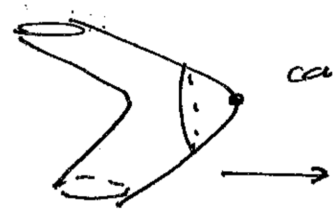
$$\Rightarrow \quad \Delta a + \partial_s (h'(e^a)) = \| \partial_s v \|^2$$

$$(\partial_s^2 + \partial_t^2) a + h''(e^a) e^a \partial_s a = \| \partial_s v \|^2 \geq 0$$

Now you can use maximum principle.

Can have

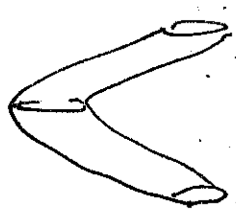
but not



can't go to the right

~~or~~

or



ok to go to the left.

Does $HF_k(H)$ depend on H?

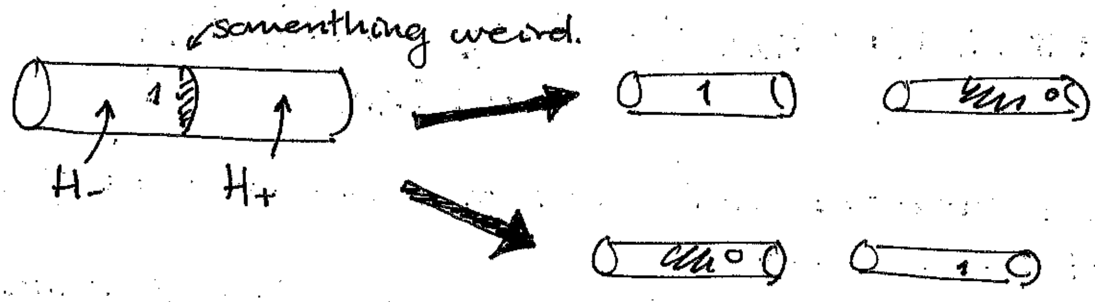
~~$HF_k(H)$ doesn't depend on J~~ does ~~...~~ depend

Suppose $H_+, H_- \in Ad(V, \lambda)$. Let $H_s : \hat{V} \to \mathbb{R}$, $H_s = H_-$ for $s << 0$ and $H_s = H_+$ for $s >> 0$. & $H_s(r,y) = h_s(r,y)$ on cylindrical end

$h_s(r,y) = a_s e^r + b_s$.

$$\Phi : CF(H_+) \longrightarrow CF(H_-)$$

$$\Phi(x_+) = \sum_{\substack{x_+ \in \mathrm{Crit}\, \mathcal{A}^H \\ |x_+| = |x_-|}} \# \mathcal{M}(x_-, x_+)\, x_-$$



$$d^{H_-} \circ \Phi = \Phi \cdot d^{H_+}$$

$$\Delta a + \underbrace{\partial_s (h_s'(e^a))}_{h_s''(e^a) e^a \partial_s a + (\partial_s h_s')(e^a)} = \| \partial v \|^2 \geq 0$$

We are in trouble unless we ask that $(\partial_s h_s') < 0$

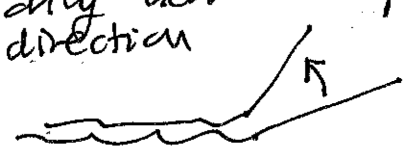
gives us space of admissible homotopies

$\Downarrow$

H_- has steeper slope than H_+

$\Downarrow$

only have map in one direction



Finally, define a partial order on $Ad(V, \lambda)$

$H_1 \prec H_2$ if $H_i < H_2$ outside a compact set.

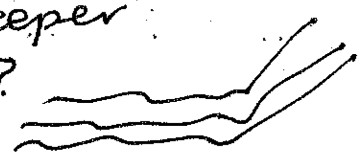
Now,
$$\begin{tikzcd} HF_k(H_1) \arrow[r] \arrow[rr, bend right] & HF_k(H_2) \arrow[r] & HF_k(H_3) \arrow[r] & \cdots \end{tikzcd}$$

we have a directed set.

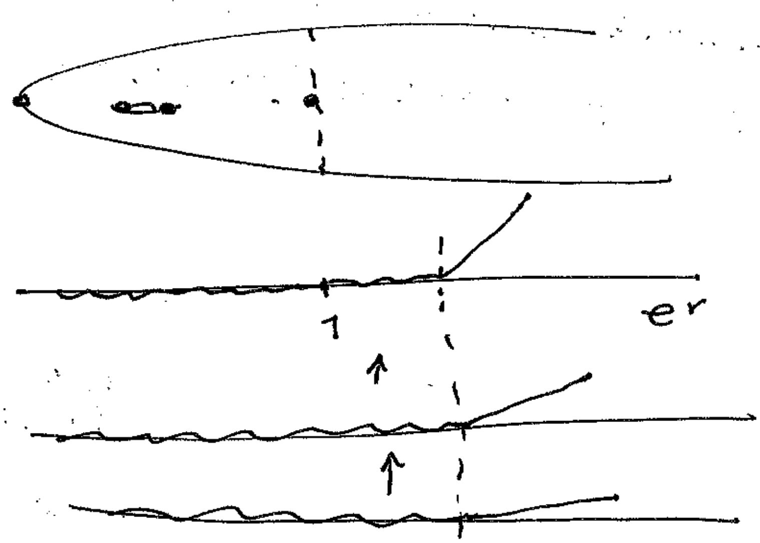
for any H, H' have $\exists \tilde{H}$
$$\begin{tikzcd} HF(H) \arrow[dr] & \\ & HF(\tilde{H}) \\ HF(H') \arrow[ur] & \end{tikzcd}$$

$$\boxed{\text{DEFINITION: } SH_k(V) = \varinjlim_{H \in Ad(V,\lambda)} HF_k(H)}$$

- It's an invariant up to exact symplectomorphism of completions.
- Doesn't measure size
- What does this limit do? making slope of Hamiltonian steeper & steeper

What are the generators of an Hamiltonian? 

Can choose cofinal sequence of hamiltonians to compute direct limit
$H_1 < H_2 < H_3 < \dots < H_k$



choose cofinal sequence