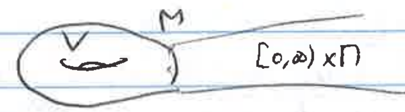
# Momchil - Symplectic homology

Goal: if $W, W'$ are two subcritical Stein fillings of $(M, \xi)$ with $c_1(W) = c_1(W') = 0$, then $H^*(W) \cong H^*(W')$.

$(V, \lambda)$, $d\lambda$ is symplectic, $Z$ st $i_Z d\lambda = \lambda$, and $Z$ is positively transverse to $\partial V \Rightarrow \alpha := \lambda|_{\partial V}$ is contact. Then, $(\hat{V}, \hat{\lambda})$ is the completion $(V,\lambda) \cup_{\partial V} ([0,\infty) \times M, e^r \alpha)$.




Pick $H : \hat{V} \to \mathbb{R}$ and define

$$A^H : C^\infty(\mathbb{R}/\mathbb{Z}, \hat{V}) \to \mathbb{R} : x \mapsto \int_{S^1} x^* \hat{\lambda} - \int_{S^1} H(x(t))\, dt.$$

Then, $dA^H_x \cdot \xi = \int_{S^1} d\lambda(\xi(t), \dot{x}(t) - X^H(x(t)))\, dt$; for $x$ to be a critical point, need $\dot{x}(t) = X^H(x(t))$.

$\Rightarrow$ crit. points are orbits of $H$.

Want to do Morse theory. To get a metric on $C^\infty(S^1, \hat{V})$, put one on $\hat{V}$ first: pick $J : T\hat{V} \to T\hat{V}$, $J^2 = -1$, $d\lambda(-, J-)$ is a Riemannian metric. For this, $\nabla_x A^H = -J(\dot{x} - X^H(x))$.

The upward gradient flow is: for a path $u : \mathbb{R}_s \times S^1 \to \hat{V}$, have $\partial_s u + J(\partial_t u - X^H) = 0$ "Floer's equation"

$\rightsquigarrow$ "Just do Morse theory": let $CF(H) := \bigoplus_{x \in crit(A^H)} \mathbb{F}_2 \cdot x$

For $c_1(V) = 0$, $CZ(x) \in \mathbb{Z}$; define $|x| = CZ(x) + n - 3$. Define a differential by $d : CF_k(H) \to CF_{k-1}(H)$:

$$d(x_+) = \sum_{\substack{x_- \in crit(A^H) \\ |x_-| = |x_+| - 1}} \#_{\mathbb{F}_2} \underbrace{\mathcal{M}(x_-, x_+)}_{\hookrightarrow \left\{ u : \begin{array}{l} \partial_s u + J(\partial_t u - X^H) = 0, \\ \lim_{s \to \pm\infty} u(s,t) = x_\pm(t) \end{array} \right\} / \mathbb{R}} x_-$$

$\mathbb{R}$-translation

For these indices, that is 0-dimensional, but we don't know whether it's finite.

Compactness: Gromov compactness works if the following holds:

(a) uniform $C^0$ bounds

(b) uniform energy bounds.

$\hookrightarrow E(u) = A^H(x_+) - A^H(x_-)$

$\Rightarrow d$ decreases action.

(b) comes for free, as $A^H(x_+)$ and $A^H(x_-)$ are fixed.

(a) needs some serious restrictions.

write $h(t) = At + B$

Definition: the spectrum $\mathrm{Spec}(\Gamma, \alpha) := \{T \in \mathbb{R} \mid \exists$ Reeb orbit of $\alpha$ with period $T\}$.

Definition: the space of admissible Hamiltonians is

$$Ad(V, \lambda) := \{H : \hat{V} \to \mathbb{R} \mid \cancel{\text{outside a compact set, } Z(H) = H \text{ and } \mathrm{slope}(H) \notin \mathrm{Spec}(\Gamma,\alpha)}$$

~~Rem. $Z = \partial_r$, so this means~~ $H : \hat{V} \to \mathbb{R}$ looks like $H(r,y) = Ae^r + b$ in the cylindrical end, and $A \notin \mathrm{Spec}(\Gamma, \alpha)$ is ~~not in~~ the slope. $\}$

Definition: a $J : T\hat{V} \to T\hat{V}$ is called cylindrical if on the cylindrical end, $J = j \oplus J_\xi$, st $j\partial_r = R$ and $J_\xi : \xi \to \xi$ is a $\mathbb{C}$-str. on $\xi$.

These choices ensures $C^0$ bounds $\Rightarrow$ compactness $\Rightarrow$ $d$ is well-defined and $d^2 = 0$.

Definition: $HF_k(H) := H_k(CF_*(H), d)$.

Rem: a priori this should depend on $J$, but in fact, it does not.

In the cylindrical end, if $u(s,t) = (a(s,t), v(s,t))$, the equation is $\Delta a + \partial_s(h'(e^a)) = \|\partial_s v\|^2 \geq 0$, so by maximum principle, we can't have max in interior:

if there is one, have $\begin{cases} \Delta a \leq 0 \text{ because max} \\ \partial_s(h'(e^a)) = h''(e^a)e^a \underbrace{\partial_s a}_{=0} \cancel{\leq} 0 \text{ because max,} \end{cases}$

hence contradiction. (or almost, since the whole thing could be $= 0$).

Independence of $H$?

$\underset{H_{s \ll 0}}{H_-}$, $\underset{H_{s \gg 0}}{H_+}$

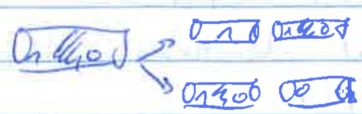
$H_+, H_- \in Ad(V, \lambda)$. Let $H_s : \hat{V} \to \mathbb{R}$ path from $H_-$ to $H_+$, and interpolate by considering solutions to the equation

$$\partial_s u + J(\partial_t u - X^{H_s}(u)) = 0$$

⚠ no $\mathbb{R}$-translation.

Counts

Define $\Phi : CF(H_+) \to CF(H_-)$, $\Phi(x_+) = \sum_{\substack{x_- \in \mathrm{Crit}(A^{H_-}) \\ |x_+| = |x_-|}} \# \mathcal{M}(x_-, x_+)\, x_-$

Possible breakings:  so it's a chain map

For compactness, we get the equation $\Delta a + \partial_s(h_s'(e^a)) = \|\partial_s v\|^2 \geq 0$, but there is a $(\partial_s h_s')(e^a)$ is there, that might ruin the previous argument. However, it's fine if $\partial_s h_s' < 0$, i.e. if $H_-$ has steeper slope than $H_+$.

So: get only a map in one direction.

→ Define a partial order on $Ad(V, \lambda)$: we say $H_1 \ll H_2$ if $H_1 < H_2$ outside a compact set (ie $\neq$ slopes, or one is shift of the other).

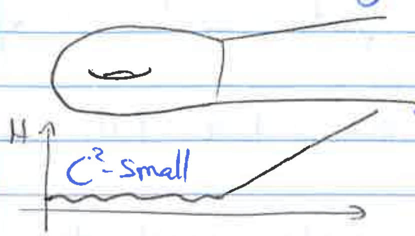
Also, by a homotopy of homotopies argument,

$$\begin{tikzcd} HF(H_1) \arrow[r] \arrow[rr, bend right] & HF(H_2) \arrow[r] & HF(H_3) \end{tikzcd}$$ commutes.

Also, this forms a directed system: $\begin{matrix} HF(H) \\ HF(H') \end{matrix} \rightrightarrows HF(\tilde{H})$

Definition. $SH_k(V) = \varinjlim_{H \in \mathcal{H}(V,\lambda)} HF_k(V)$; can compute it using a cofinal sequence.

$\leadsto$ What are the generators? Take a cofinal sequence of Hamiltonians looking like this, with increasing slope in the cylindrical end.




Fact 1: if $H$ is $C^2$-small ($\|\mathrm{Hess}\, H\| \ll \varepsilon$), then all the 1-periodic orbits of $X^H$ are constant, ie crit pts of $H$. Moreover, the Floer trajectories connecting crit pts of $H$ are just Morse flow lines.

Compute: $X^H = h'(e^r) R$, so orbits lie in constant $r$-level, and they coincide with Reeb orbits, up to reparametrization, of period $h'(e^{r_0}) = T$, and has action $A^H(x) = e^{r_0} T - h(e^{r_0})$
$\geq T - h(e^{r_0}) > 0$

~~There is~~ So, due to the hypothesis on slope & spectrum, we see that there are no Reeb orbits in the linear part.

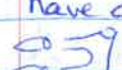
Also, there is $0 \to \overbrace{CF^{<0}(H)}^{\text{crit pts}} \to CF(H) \to \overbrace{CF(H)/CF^{<0}(H)}^{\text{Reeb orbits}} \to 0$,
because the differential decreases action. So, get exact triangle

$$SH^{<0}(V) = H^{-k}(V) \longrightarrow SH_k(V) \longrightarrow SH_k^+(V) \longrightarrow$$

[Bourgeois - Oancea]

Theorem: if all Reeb orbits of $(\Pi, \alpha)$ satisfy $CZ(\gamma) + n - 3 > 0$ and $V, W$ are 2 exact fillings of $\Pi$ with $c_1(V) = c_1(W) = 0$, then $SH^+(V) \simeq SH^+(W)$.
So, it's an invariant of the compact manifold.

Proof: we show no cylinder goes into the filling. Stretch the neck: if we have a seq. of cylinders going in the filling, stretch the neck $\Rightarrow$ it has to break in



or

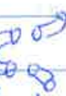
, ruled out by index: it will be negative. $\square$

with $c_1(\text{filling})=0$

Theorem (M-L Yau) if $M$ is subcritically Stein fillable, then $M$ admits a contact form st $CZ(\gamma)+n-3>0$ $\forall$ Reeb orbit $\gamma$.

Theorem. (Cieliebak) $V$ subcritical Stein with $c_1(V)=0 \Rightarrow SH(V)=0$.

So, we can prove the goal: $SH(V)=SH(W)=0$, and $SH^+(V)=SH^+(W)$, so by the exact triangle above, we have $H^*(V)=H^*(W)$. The index assumption on Reeb orbits is satisfied by Yau's theorem.

Rem: Floer cylinders have to approach the negative orbit "by the right a little bit": $\exists\,(s_0,t_0)\in\mathbb{R}\times S^1$ st $a(s_0,t_0)>r_-$, where $r_-$ is the r-level of the negative Reeb orbit.
This is due to Bourgeois-Oancea, and helps to rule out  above: must have  for the bottom right one, but that contradicts the maximum principle.
One should remember this as one of the fundamental properties of holomorphic curves, along with the maximum principle. (Kyler)