[FLEXIBLE WEINSTEIN STRUCTURES - Ziva]

Outline:

I. Intro to FWS
II. Loose legendrians
III. Classification of FWS
IV. Examples.

Weinstein mfld:

$(W^{2n}, \omega, X, \phi)$

W^{2n}: open mfld

ω: symp form

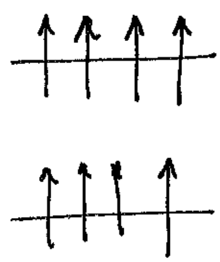
X: Liaville v.f. for ω, gradient like for ϕ ($X = 0$ @ crit points of ϕ & ~~positive~~ $d\phi(X) > 0$ away from crit. points.)

ϕ: exhausting generalized Morse function. $W \to \mathbb{R}$. proper + bdd below; critical points nondegenerate or embryonic $\langle\ \rangle$

Weinstein Cobordism

W with $\partial W = \partial_- W \sqcup \partial_+ W$

bandary = regular level sets of ϕ.



$\partial_\pm W$ reg. level sets for ϕ

Weinstein domain $\partial_- W = \emptyset$.

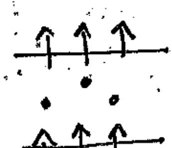
NOTES:

- $\omega = d\lambda \Rightarrow \forall$ regular $c \in \mathbb{R}$, $\alpha_c = \lambda|_{\phi^{-1}(c)}$ $\xi_c = \ker \alpha_c$
- $\forall\ p \in \mathrm{crit}(\phi)$, W_p^- is isotropic (symp) $W_p^- \cap \phi^{-1}(c)$ isotropic (contact) $\Rightarrow p \in \mathrm{Crit}(\phi)$, $\mathrm{ind}(p) \leq n$.

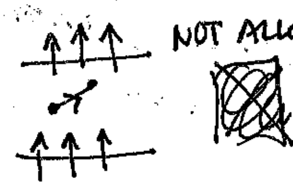
FLEXIBILITY:

- Cut W along regular level sets into elementary W cobordisms.

no trayectories btwn critical pts.



NOT ALLOWED!



($n \geq 2$)
DEFN: An elementary Weinstein cobordism is flexible if attaching spheres of all index n-handles form a loose legendrian (link) in $\partial_- W$.
Where W is flexible if it can be decomposed with into flex elementary cobordisms.

Notes: subcritical $\Rightarrow$ flexible
($n=2$) flexible $\Leftrightarrow$ subcritical or $\xi_{\partial_- W}$ is OT.

LOOSE LEGENDRIANS

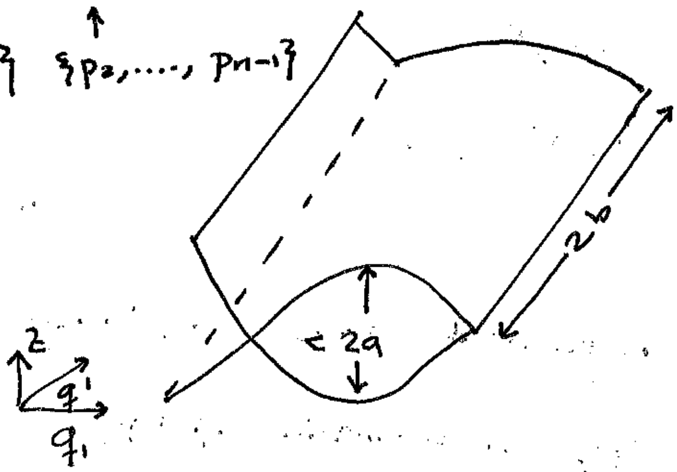
$\swarrow$ Local model $\qquad$ $\searrow$ satisfy an h-principle

on $(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i)$

$$R_{abc} = \left\{ \begin{array}{l} |q_1|, |p_1| \leq 1, |z| \leq a \\ |q'| \leq b, \ |p'| \leq c \end{array} \right\}$$

$q' = \{q_2, \dots, q_{n-1}\}$, $p' = \{p_2, \dots, p_{n-1}\}$

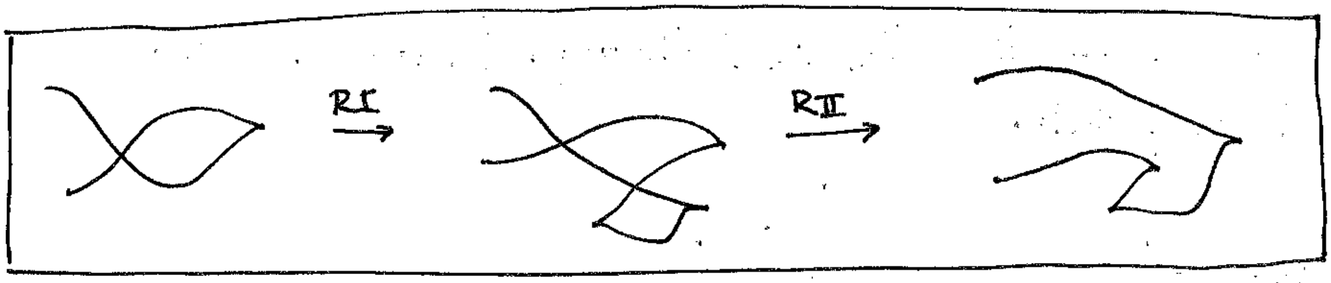
Λ_0. cusp $\times$ disk



DEFN: $(R_{a,b,c}, \Lambda_0)$ is standard loose legendrian chart if $a < b\ c$

DEFN: $\Lambda \subset (M^{2n-1}, \xi)$ connected legendrian is loose if $\exists$ Darboux chart U s.t. $(U, U \cap \Lambda) \simeq (R_{abc}, \Lambda_0)$.

Note that we have no other legendrians intersecting in that Darboux chart.



Loose legendrians satisfy an h-principle:

FORMAL LEGENDRIAN

$$F^s : T\Lambda \longrightarrow TM \quad \text{homotopy of bundle homomorphisms } s \in [0,1]$$
$$\downarrow$$
$$f : \Lambda^n \hookrightarrow (M^{2n-1}, \xi) \quad \text{smooth embedding}$$

If $F^0 = df$ & F has a lag image in ξ then (f, F^s) is a formal legendrian.

Note: If leg embedding $f : \Lambda \hookrightarrow M$ is formal
$(f, F^s \equiv df)$ $\quad (f_t, F^s_t)$ $t \in [0,1]$ formal legendrian isotopy.

Existence: Given $(f, F^s) \hookrightarrow (M, \xi)$ is it formally isotopic to a legendrian?

UNIQUENESS: Given (f_t, F^s_t) btwn f_0, f_1 legendrian is there a leg iso btwn them?

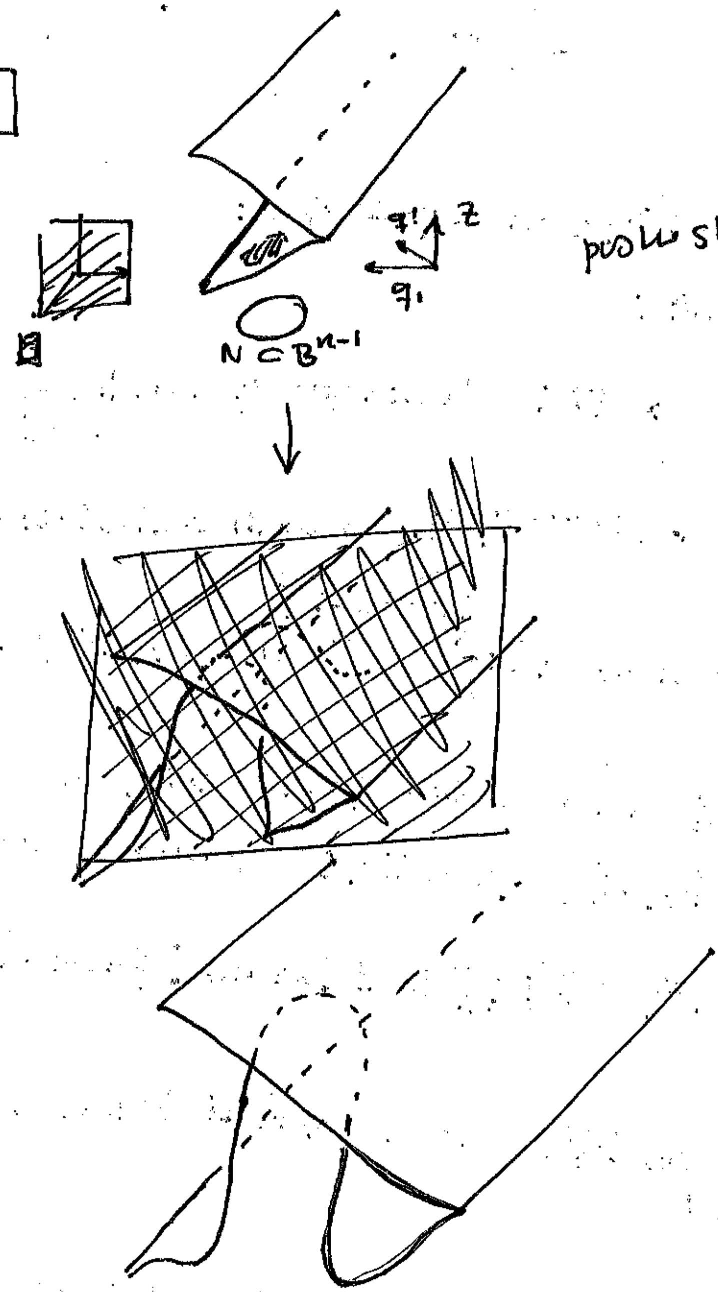
THM (Murphy) h-principle for loose legendrians

$(M^{2n-1 \geq 5}, \xi)$ 1) Given a $(f, F^s) \hookrightarrow (M, \xi)$ of Λ^n
$\exists$ loose legendrian embedding
$$\hat{f} : \Lambda \hookrightarrow M$$
that is C^0-close to f & formally legendrian isotopic to (f, F^s)

(!) Given (f_t, F^s_t) $f_0, f_1 : \Lambda \hookrightarrow M$ loose legendrian embeddings, $\exists$ leg iso $\hat{f}_t$ $(\hat{f}_0 = f_0, \hat{f}_1 = f_1)$ C^0 close to f_t

and homotopic to (f_t, F_t^s) through formal leg iso w/ fixed endpoints.

$\boxed{\text{Stabilization}}$



push shaded region up.

PROP (MURPHY)

Λ_1 loose

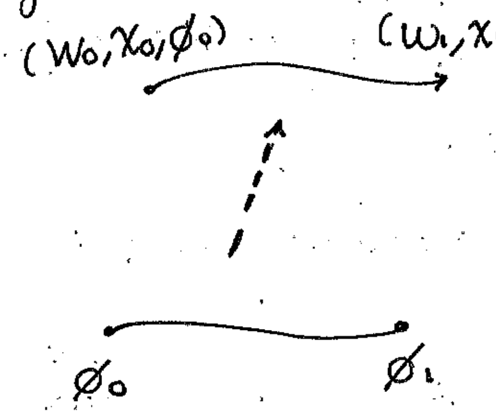
$\chi(N) = 0 \Rightarrow \Lambda_1$ is formally leg iso to Λ_0

THM: (Cieliebak-Eliashberg)

$\mathcal{M} \neq \emptyset$ homotopy class of non degenerate 2-forms on $W^{2n>4}$ (domain or mfld) then

$$\begin{array}{ccc} \mathcal{M}: \text{Weinstein}^{flex}_{\mathcal{M}} & & (\omega, X, \phi) \\ \downarrow & & \downarrow \\ \text{Morse} & & \phi \end{array}$$

is surjective, has path connected fibers & has path lifting property.



Morse n = crit. points w/ index < n

CONJECTURE $\mathcal{M}$ is a Serre fibration (homotopy lifting property on closed disks), w/ contractible fibers.

THM (C-E) Weinstein h-cobordism

Any flexible Weinstein structure on $W^{2n-4} = Y \times [0,1]$ is homotopic to a Weinstein structure (W, ω, X, ϕ) where $\phi: W \to [0,1]$ has no critical points.

(symplectic product cobordism).

THM: (C-E) $(W_1, \omega_1, X_1, \phi_1)$ & $(W_2, \omega_2, X_2, \phi_2)$ flexible W. structures $f: W_1 \to W_2$ diffeo s.t. $f^* TW_2 \cong TW_1$ (as symplectic vector bundles.) Then f is isotopic to a symplectomorphism

EXAMPLES:

□ THM: (Casals-Murphy)

$$\mathcal{X}^n_{1,b} = \{ (x,y,z) \mid xy^b + \sum_{i=1}^{n-1} z_i^2 = 1 \} \subseteq \mathbb{C}^{n+1}$$

family of flexible Weinstein mflds $\forall$ $b \geq 2$.

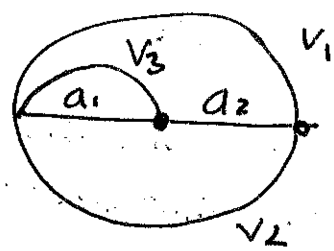
(attach a crit. Weinst handle to $\mathbb{B}^{2n}$ along a cusp spinning torus knot)

(Harris, Medanski, Seidel, Casals-Murphy ...)

□ THM: The Weinstein 6-fold (E, λ, ϕ) that is not flexible but it embeds as a Weinstein sublevel set into the unique flex W. structure $(T^*S^3, \lambda_f, \phi_f)$

$$\begin{array}{ccc} \mathbb{E} = \{ (x,y,z,w) & \mid x(xy-1) = z^2+w^2 \} \subseteq \mathbb{C}^4 \\ \pi \downarrow & \downarrow \\ \mathbb{C} & (x+y) \end{array}$$

NOT flexible. but once its embeded it becomes flexible.



Lefschetz diagram