Ziva - Flexible Weinstein structures

Outline:
1) Intro to flexible Weinstein structures
2) Loose Legendrians
3) Classification of FWS
4) Examples

1) **Weinstein manifold**: $(W^{2n}, \omega, X, \phi)$: $n \geq 2$
- W open manifold
- ω symplectic form
- X Liouville for ω, gradient-like for ϕ
- ϕ exhausting generalized Morse $W \to \mathbb{R}$

"Exhausting" = proper and bounded below; "generalized" = critical points are non-degenerate or embryonic, "gradient-like" means that $X=0$ at $\mathrm{crit}(\phi)$ and $d\phi(X) > 0$ away from $\mathrm{crit}(\phi)$.

Weinstein cobordism: W with $\partial W = \partial_- W \sqcup \partial_+ W$, such that $\partial_\pm W$ are regular level sets for ϕ.

Weinstein domain: if $\partial_- W = \emptyset$.

$\partial_+ W$

Notes: * X Liouville $\Rightarrow \omega = d\lambda \Rightarrow \forall c \in \mathbb{R}$ regular for ϕ, $\alpha_c := \lambda|_{\phi^{-1}(c)}$ is contact on $\phi^{-1}(c)$.

* $\forall p \in \mathrm{crit}(\phi)$, W_p^- is isotropic (symplectically), and $W_p^- \cap \phi^{-1}(c)$ for c regular is isotropic (contactically)

$\Rightarrow$ for $p \in \mathrm{crit}(\phi)$, $\mathrm{ind}(p) \leq n$.

Flexibility: - cut W along regular level sets into elementary W. cobordisms, ie in each of them, there is no trajectory between critical points.

Definition: an elementary Weinstein cobordism is *flexible* if the attaching spheres of all index n handles form a loose Legendrian link in $\partial_- W$. The whole cobordism is flexible if it can be decomposed into flexible elementary cobordisms.

Rem: * subcritical $\Rightarrow$ flexible

* for $n=2$, extend by defining: W subcritical or $\xi_{\partial_- W}$ is overtwisted.

2) **Loose Legendrians**:
- local model
- satisfy an h-principle.

Local model: on $(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i dq_i)$, define
$R_{abc} := \{ |q_1|, |p_1| \leq 1, |z| \leq a, |q'| \leq b, |p'| \leq c \}$
$q' = (q_2, \dots, q_{n-1})$, $p' = (p_2, \dots, p_{n-1})$

Define Λ_0:  in R_{abc} (in slice $p=0$)

Definition: (R_{abc}, Λ_0) is a standard loose Legendrian chart if $a<bc$.
Definition: a connected Legendrian $\Lambda \subseteq (\Gamma^{2n-1}, \xi)$ is loose if $\exists$ Darboux chart $\mathcal{U}$ such that $(\mathcal{U}, \mathcal{U} \cap \Lambda) \cong (R_{abc}, \Lambda_0)$.

Rem: , so having a fish is equivalent to having a zigzag.

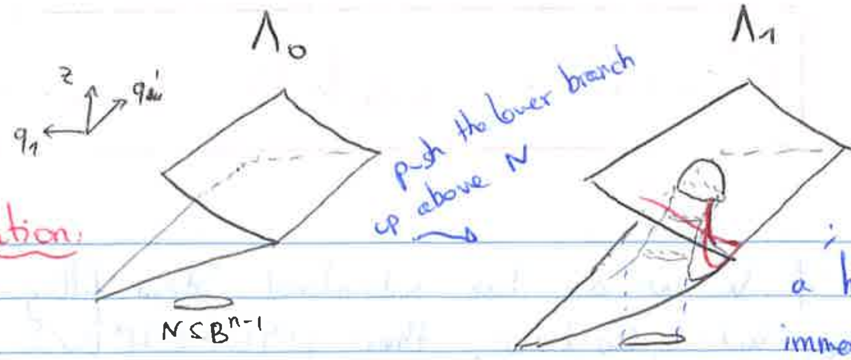
Rem: the point is also that there is only Λ_0 in that Darboux chart, and not other piece. So, one could have a link composed of a loose Legendrians that it not loose.

Formal Legendrians: $F^s : T\Lambda \to TM$ homotopy of bundle monomorphisms
$\downarrow$
$f : \Lambda^n \hookrightarrow (\Gamma^{2n-1}, \xi)$ smooth embedding.
If $F^0 = df$ and F^1 has a Lagrangian image in ξ, then (f, F^s) is a formal Legendrian.
Rem: a Legendrian embedding $f : \Lambda \hookrightarrow M$ is formal: take $F^s = df$ $\forall s \in [0,1]$.
Definition: a formal Legendrian isotopy is a family (f_t, F_t^s) of formal Legendrians.

Existence: given $(f, F^s) \hookrightarrow (M, \xi)$, is it formally isotopic to a Legendrian?
Uniqueness: given (f_t, F_t^s) between f_0, f_1 Legendrian, is there a Legendrian isotopy between them?

Theorem (Murphy) "h-principle for loose Legendrians": $(M^{2n-1 \geq 5}, \xi)$
(E) Given $(f, F^s) \hookrightarrow (M, \xi)$ of Λ^n, $\exists$ loose Legendrian $\hat{f} : \Lambda \hookrightarrow M$ C^0-close to f and formally Leg. isotopic to (f, F^s)
(U) Given (f_t, F_t^s) between $f_0, f_1 : \Lambda \hookrightarrow M$ loose Leg. embeddings, there is a Legendrian isotopy $\hat{f}_t$ from $\hat{f}_0 = f_0$ and $\hat{f}_1 = f_1$, C^0-close to f_t and homotopic to it through formal Leg. isotopies with fixed endpoints.

Stabilization:



; this is doable as a homotopy through immersed Legendrians.

Proposition (Murphy) * Λ_1 is loose
* $\chi(N)=0 \Rightarrow \Lambda_1$ is formally leg. isotopic to Λ_0.

Idea for Λ_1 loose: see the red fish above.

3) **Theorem** (Cieliebak–Eliashberg) if $\phi \neq \eta$ = homotopy classes of non-degenerate 2-forms on $W^{2n>4}$ (domain, or rel ∂), then
$\mathcal{M}: \text{Weinstein}^{flex}_{\eta}(\omega, X, \phi) \to \text{Morse}_n : (\omega, X, \phi) \mapsto \phi$ is surjective, has path-connected fiber, and has the path-lifting property.
Conjecture: $\mathcal{M}$ is a Serre fibration, with contractible fibers.

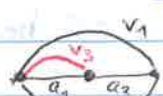
Theorem (Cieliebak–Eliashberg) "Weinstein h-cobordism": Any flexible Weinstein structure on $W^{2n>4} = Y \times [0,1]$ is homotopic to a Weinstein structure (W, ω, X, ϕ) where $\phi: W \to [0,1]$ has no critical points.

Theorem (Cieliebak–Eliashberg) $(W_i, \omega_i, X_i, \phi_i)$ $(i=1,2)$ two flexible Weinstein structures with $f: W_1 \to W_2$ diffeo st $f^*TW_2 = TW_1$ as sympl. vector bundles, then f is isotopic to a symplectomorphism.
⚠ f compactly supported $\not\Rightarrow$ symplectomorphism compactly supported.

4) **Theorem** (Casals–Murphy) $X^n_{1,b} = \{(x,y,\underline{z}) \mid xy^b + \sum_{i=1}^{n-1} z_i^2 = 1\} \subseteq \mathbb{C}^{n+1}$ are flexible $\forall\, b \geq 2$.

Theorem: there exists a Weinstein 6-fold (E, λ, ϕ) which is not flexible, but it embeds as a Weinstein sublevel set into the unique flexible Weinstein structure (T^*S^3, λ, ϕ)

Construction: $E = \{(x,y,z,w) \mid x(xy-1) = z^2+w^2\} \subseteq \mathbb{C}^4$
$\pi \downarrow \quad\quad \downarrow$
$\mathbb{C} \quad\quad x+y$



Rem: the h-principle for loose Legendrians is true parametrically, if we fix the loose chart. But it's not written anywhere.