Danny: TIGHT CONTACT STRUCTURES AND SEIBERG-WITTEN EQNS

DANI

Goal: ~~[illegible]~~ distinguish tight contact structures ~~that are homotopic as plane~~ on M^3, closed.

ξ_1 & ξ_2 are homotopic if underlying plane fields are homotopic (thru any old plane field)

ξ_1 & ξ_2 are isotopic if they are homotopic thru contact structures. (can be joined by a path ξ_t of contact structures)

ξ_1 & ξ_2 are isomorphic if the contact mflds are contactomorphic.

$\boxed{\text{homotopic} \not\Rightarrow \text{isomorphic.}}$

REMARKS: 1. isomorphic $\not\Rightarrow$ homotopic b/c underlying diffeo need not be isotopic to identity.

2. isomorphic + homotopic $\not\Rightarrow$ isotopic

3. isotopic $\Rightarrow$ homotopic + isomorphic.

4. $\mathrm{Cont}(M) = \mathrm{Tight}(M) \sqcup OT(M)$

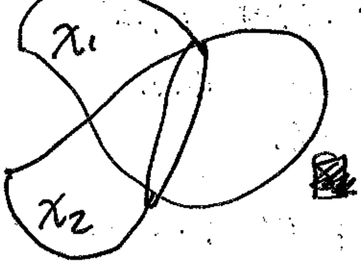
For $\xi_1, \xi_2 \in OT(M)$ homotopic $\Leftrightarrow$ isotopic
"they satisfy the h-principle."

5. Gray $\Rightarrow \exists\, \phi_t \in \mathrm{Diff}$ $\phi_t^* \xi_1 = \xi_t$ rmk 5 $\Rightarrow$ rmk 3

THM (Lisca-Matic) $\forall\, n \geq 0$, there exists a homology sphere with $\geq n$ contact structures which are homotopic but not isomorphic.

Build these structures as diff legendrian realizations of the same Kirby diagram.

If contact structures are isomorphic.



$S_1 \xrightarrow{s} S_2$

$c_1(X) \neq c_2(X)$.

§ 2. Legendrian link $K \hookrightarrow (S^3, \xi_{std})$ legendrian

$tb(K) = \# K' \cap S = \# \times + \# \times + \# \times - \# \times - \# \boxed{} >$

K' = Reeb pushoff S = Seifert surface

$(S \hookrightarrow S^3, \partial S = K)$

$r(K)$ = rotation # of TK w/ respect to a trivilization of $\xi|_S$

$= \frac{1}{2}(\# \succ + \# \prec - \# \succ - \# \prec)$

$L = \bigsqcup_i K_i \subset S^3$

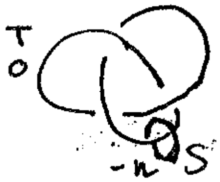
surgery $W = B^4 \cup_i H_i^2$ Stein structure.

framing imposed on K_i is $tb(K_i) - 1$.

$\underbrace{\langle \underset{\substack{\downarrow \\ \text{obstruction to} \\ \text{extending a} \\ \text{trivialization}}}{c_1(W)}, [H_i^2] \rangle}_{\text{obstruction to extending trivialization to handle.}} = r(K_i)$.

$\mathbb{C}^6$

$N_n = B^4 \cup_i H_i$



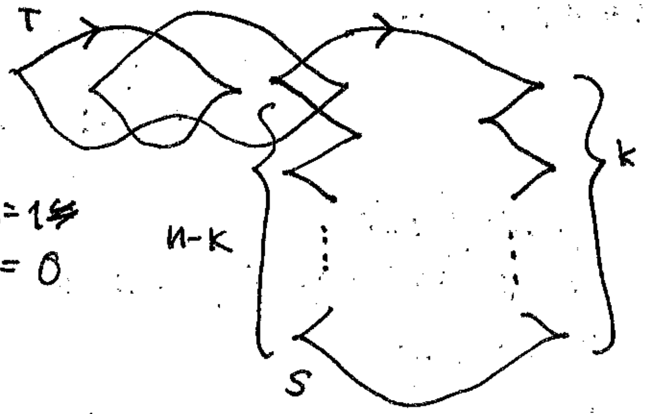
Homology sphere.

$= \{x^2 + y^3 + z^{6n-1} = 0\} \cap S^5$

$\partial N_n = \Sigma(2,3,6n-1)$

Brieskorn sphere

W_n^k Stein structure on N_n for each $1 \le k \le n-1$.



$tb = 1$
$r = 0$

$tb = -n+1$
$r = 2k-n$.

$$c_1(W_n^k) = (2k-n)\,PD(T)$$

$$\langle c_1(\omega), T\rangle = 0$$

$$\langle c_1(\omega), S\rangle = 2k-n$$

ξ_n^k contact structure on $\partial W^k \underset{\text{top}}{=} \partial N_n$

§3. <u>LEMMA</u>: X_i^4 are almost complex mflds with boundary & integral homology spheres. $\partial X_1 \underset{\text{diffeo}}{\simeq} \partial X_2$. Let ξ_i be the plane field on $T\partial X_i$ formed by the complex tangencies $\xi_i = T\partial X_i \cap J_i(T\partial X_i) \subset T\partial X_i$

Then ξ_1 & ξ_2 are homotopic as plane fields $\Leftrightarrow$

$$c_1^2(X_1) - 2\chi(X_1) - 3\sigma(X_1) = c_1^2(X_2) - 2\chi(X_2) - 3\sigma(X_2).$$

Sketch: you look at the clutching function on each plane use fact that you have a homology sphere.

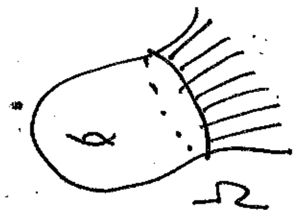
COROLLARY $\xi_n^k \simeq \xi_n^{k'}$ as plane fields

Proof: $X_1 \simeq X_2$ as a smooth manifold. Need to check

$$c_1^2(W_n^k) = c_1^2(W_n^{k'}).$$

$$c_1(W_n^k) = (2k-n)\,PD(T)$$

$PD(T)\cdot PD(T) = 0$ by framing of T.



§4 reference: hot tub seminar.

A compact complex mfld has no nonconst holomorphic functions
Stein mflds is noncompact & has many holomorphic functions
has convexity property $\Rightarrow \bar\partial$ eqn is solvable

$\Rightarrow$ find lots of holomorphic functions $\Rightarrow \underset{\text{stein}}{\Omega} \hookrightarrow \mathbb{C}^N$
proper holomorphic embedding

LEMMA: Let Ω be a stein domain. Then Ω admits a holomorphic embedding into a compact Kähler surface S satisfying

(1) minimal
(2) general type
(3) $b_+^2(S) > 1$

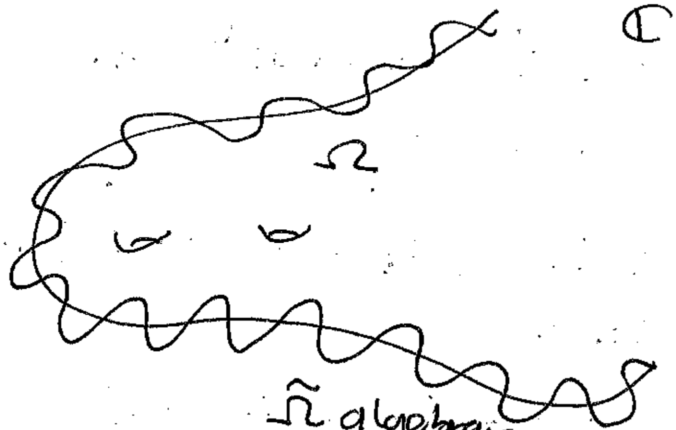
Proof: (1) minimal $\rightsquigarrow$ there are no (exceptional) curves $E \subseteq S$ s.t.
$E\cdot E = -1 \quad E \simeq \mathbb{C}P^1$
(cannot blow S down)

(2) general type: Kodaira dim $= 2$ (maximal)
Generic condition (think of it as negative curvature)
$\Rightarrow K\cdot[\omega] > 0$ & $K\cdot K > 0$.

(3) $b_+^2(S) =$ rk of definite positive part of H^2.

"Proof":

$\mathbb{C}^n$



$\tilde{\Omega}$ algebraic
cut out by polynomials

Let $S = \hat{\tilde{\Omega}} \subseteq \mathbb{CP}^n$ compactification given by homogeneizing the polynomials cutting out $\tilde{\Omega}$.

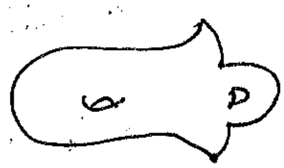
Then, resolve singularities or avoid them.

By adjunction if ~~powers~~ degrees of polynomials are high $\Rightarrow$ S has ample K $\Rightarrow$ minimal of general type.

$\hookrightarrow$ Still not done, for condition 3)

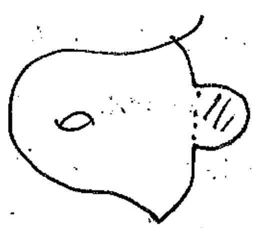
$\Omega \subset \Omega'$

$b_+^2(\Omega') > 1$



repeat ......

$b_+^2(N_n) = 1 \Rightarrow$



H_2 splits if we have boundary is a homology sphere.

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5. Seiberg-Witten theory

a Map $SW: Spin^{\mathbb{C}}(M^4) \longrightarrow \mathbb{Z}$

$SW(\mathfrak{s})$ = # of solns to SW eqns

Call $c \in H^2(M)$ Basic if $\exists\ S \in \mathrm{Spin}^{c}$ s.t. $\quad K = \Lambda^2 T^*M = -c_1(M)$

$SW(S) \neq 0$ & $c_1(\alpha) = c$.

$\Rightarrow$ If $b_+^2 > 1$, $\mathrm{Basic}(M) = \{c \mid c \text{ is basic}\}$ is a diffeomorphism invariant.

(SAW) If M is symplectic $\Rightarrow \pm c_1(M) \in \mathrm{Basic}(M)$

FACT: If M is compact Kähler minimal of general type
$\Rightarrow \pm c_1(M)$ is the only basic class.

IDEA: (1) minimal + general type $\quad K \cdot K > 0$

$K \cdot [\omega] > 0$

(2) Assume $c \in \mathrm{Basic}(M)$ Then

$\dim(\mathcal{M}_c) \geq 0 = \dim(\mathcal{M}_\omega)$.

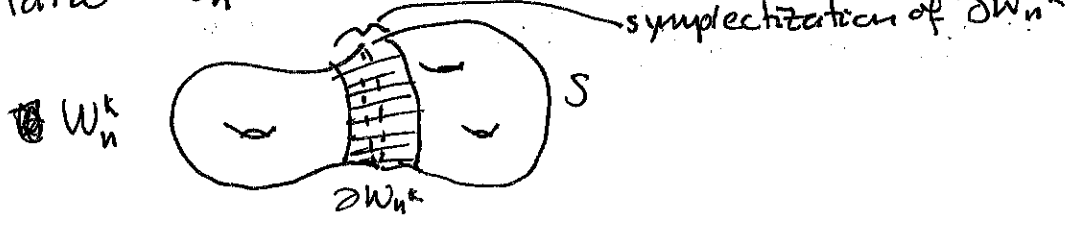
$$c^2 - 2\chi - 3\sigma \geq K^2 - 2\chi - 3\sigma$$

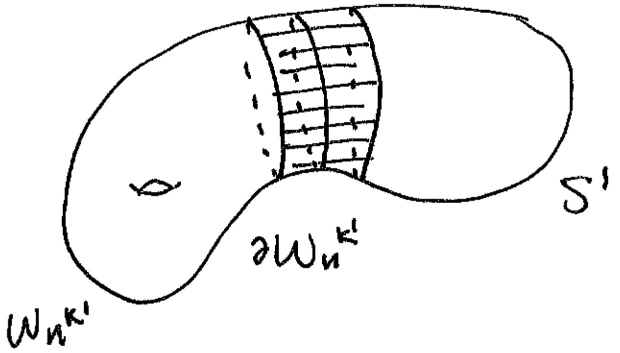
$$\therefore \boxed{c^2 \geq K^2}$$

(3) Kähler condition $\Rightarrow$ Hodge theory puts huge restrictions on top of mfld
$\Rightarrow$ Hodge index theorem can be used to play (1) & (2) against each other to derive a contradiction unless $c = \pm K$.

§6 THM $\xi_n^k \sim \xi_n^{k'}$ isomorphic $\Leftrightarrow$ $K = k'$ or $k = n - k'$

Proof: Take $W_n^k \subseteq S$ as in the lemma





If we had glued in $W_n^{k'}$ instead

Gompf: Every diffeomorphism of the Brieskorn sphere $\phi \in \mathrm{Diff}(\partial N_n)$ extends to N_n & acts as ± 1 on H^*

(1) isotopy $\longleftrightarrow$ homotopy for diff for

(2) algebraic topology $\qquad \mathrm{Diff}(S^3) = SO(4)$

Now, $id : S \setminus W_n^k \longrightarrow S' \setminus W_n^{k'}$ extends to a

$\phi : S \longrightarrow S'$ diffeo.

Because $\phi^* \mathrm{Basic}(S') = \mathrm{Basic}(S)$

$$\phi^* \{\pm c_1(S')\} = \{\pm c_1(S)\}$$

$$\phi^* c_1(W_n^k) = \pm c_1(W_n^{k'})$$

$$\pm c_1(W_n^k) = [illegible]$$

$$\pm(2k-n)\,PD(T) = \pm(2k'-n)\,PD(T) \qquad \square$$

[illegible]