Dani - Tight contact structures and Seiberg-Witten equations

Plan: 1) The goal
2) The construction
3) Some algebraic topology
4) Some Stein geometry
5) Some Seiberg-Witten theory
6) The proof.

1) Goal: distinguish contact structures on M^3 closed. Two contact structures ξ_1, ξ_2 are homotopic if the underlying plane fields are. They are isotopic if they ~~are~~ can be joined by a path ξ_t of actual contact structures. They are isomorphic if (M^3, ξ_1) and (M^3, ξ_2) are contactomorphic. We will show: homotopic $\not\Rightarrow$ isomorphic.
Rem: isomorphic $\not\Rightarrow$ homotopic: the diffeo need to be isotopic to the identity.
Rem: isomorphic + homotopic $\not\Rightarrow$ isotopic
Rem: isotopic $\Rightarrow$ homotopic + isomorphic.
Rem: $\mathrm{Cont}(M) = \mathrm{Tight}(M) \amalg \mathrm{OT}(M)$; OT means: contains (OT disk).
Theorem: for $\xi_1, \xi_2 \in \mathrm{OT}(M)$, homotopic $\Leftrightarrow$ isotopic.
Rem: Gray $\Rightarrow$ if ξ_t family of contact structures, $\exists f_t \in \mathrm{Diff}$: $f_t^* \xi_t = \xi_1$.

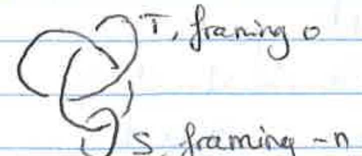
Theorem: [Lisca-Matić] $\forall n \geq 0$, $\exists$ homology sphere with at least n contact structures which are homotopic but not isomorphic.

We will build them as different Leg. realizations of the same Kirby diagram.
We'll assume $c_1(X_1) \neq c_1(X_2)$ but X_1, X_2 with same boundary ; SW will imply they have the same c_1 $\rightsquigarrow$ contradiction.

2) $K \hookrightarrow (S^3, \xi_{std})$ Legendrian.
$\rightarrow$ $tb(K) = \# K' \cap S$, where K' = Reeb pushoff and S = Seifert surface.
$\rightarrow$ $r(K)$ = rot# of TK wrt a trivialization of $\xi|_S$.
Rem: $tb(K) = \# X + \# X - \# X - \# X - \# >$ (inv. of framed knots)
$r(K) = \frac{1}{2}(\# \supset + \# \subset - \# \supset - \# \subset)$ (inv. of never vertical knots)
in the front projection.

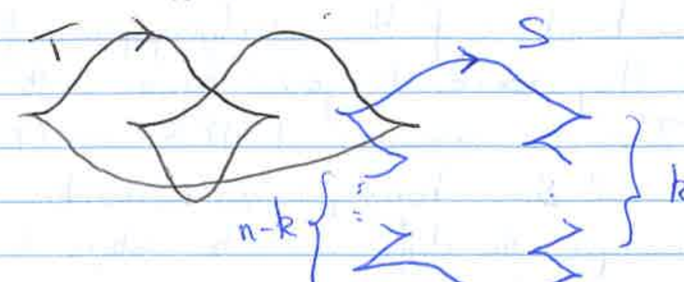
Let $L = \amalg_i K_i \subseteq S^3$ and surgery $\rightsquigarrow W = B^4 \cup_i H_i^2$ has a Stein structure (framing on K_i is $tb - 1$). Furthermore,
$\langle c_1(W), [H_i^2] \rangle = r(K_i)$: think of c_1 as trivialization to

extending over the handle, this is also what rot. is.

Let $N_n = B^4 \cup_i H_i$, 

, turns out that $\partial N_n = \underset{\text{Brieskorn sphere}}{\Sigma(2,3,6n-1)} = \{x^2+y^3+z^{6n-1}=0\} \cap S^5 \subseteq \mathbb{C}^3$.

We have W_n^k Stein structure on N_n for each $1 \leq k \leq n-1$:



$tb(S) = -n+1$
$r(S) = 2k-n$
$tb(T) = -1$
$r(T) = 0$

<u>Rem:</u> By Eliashberg - Fraser, this realizes all unknots with $tb = -n+1$.

Also, $c_1(W_n^k) = (2k-n)PD(T)$. Indeed, we have

$\langle c_1(W), \text{handle corresp. to } T\rangle = 0$

$\langle c_1(W), \text{handle corresp. to } S\rangle = 2k-n$

And $H^2(W_n^k)$ is free of rk 2 gen. by T and S, so we must have.

So, we have ξ_n^k contact structures on $\partial W_n^k \overset{\text{topologically}}{=} \partial N_n$.

3) <u>Lemma (Gompf)</u>: X_i^4 are integral homology spheres almost-$\mathbb{C}$ manifolds with boundaries ∂X_1 and ∂X_2 diffeomorphic. Let ξ_i be the plane field on $T\partial X_i$ formed by the complex tangencies $\xi_i := T\partial X_i \cap J_i(T\partial X_i) \subseteq T\partial X_i$. (these are just plane fields, maybe not contact). Then,

ξ_1 and ξ_2 are homotopic as plane fields

$$\Updownarrow$$

$$c_1^2(X_1) - 3\sigma(X_1) - 2\chi(X_1) = c_1^2(X_2) - 2\chi(X_2) - 3\sigma(X_2).$$

<u>"Proof:"</u> compute difference between ξ_1 and ξ_2 in terms of clutching function, then in terms of Pontrjagin numbers, then use Hirzebruch signature formula. "□"

<u>Corollary</u>: $\xi_n^k \simeq \xi_n^{k'}$ as plane fields.

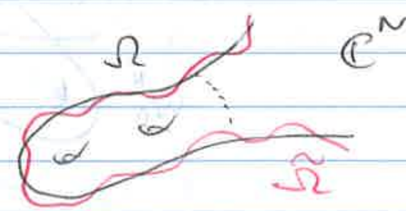
same σ and χ.

Proof: $X_1 \simeq X_2$ as smooth manifolds. Need to check $c_1^2(W_n^k) = c_1^2(W_n^k)$, but $c_1(W_n^k) = (2k-n)\,PD(T)$, and $PD(T)\cdot PD(T) = 0$ because that is the framing of T. $\square$

4) Ref: hot tub seminar. Compact manifolds have no non-cst hol. function; Stein manifolds have plenty.
Stein: convexity $\Rightarrow$ $\bar\partial$ equation is solvable $\Rightarrow$ find lots of hol. fcts $\Rightarrow$ can separate points $\Rightarrow$ can find proper hol. embedding $\Omega \to \mathbb{C}^N$.
A Stein domain is "the compact part of a Stein manifold.
Lemma: Let Ω be a Stein domain. Then, Ω admits a holomorphic embedding into a compact Kähler surface S satisfying
(1) minimal (2) general type (3) $b_+^2(S) > 1$.

(1)	(2)	(3)
no $E \subseteq S$ with $E \simeq \mathbb{CP}^1$ and $E\cdot E = -1$, ie we can't blow S down.	Kodaira dim $= 2$ (maximal). This is a generic condition (think of it as neg. curvature)	b_+^2 = pos. definite part of H^2, wrt the intersection form.

Rem: general type $\Rightarrow$ $K\cdot[\omega] > 0$ and $K\cdot K > 0$.



Proof: embed it in $\mathbb{C}^N$; approximate it by one given by polynomials (Stein $\Rightarrow$ we can): $\tilde\Omega$ algebraic. Let $S = \hat{\tilde\Omega} \subseteq \mathbb{CP}^N$ the compactification given by homogeneizing the polynomials cutting out $\tilde\Omega$; then, resolve singularities. By adjunction, if the ~~power~~ degrees of polynomials are high, then S has ample canonical bundle $\Rightarrow$ minimal of general type.
For (3), we modify it slightly: take original Ω; if b_2^+ is not large enough, do a surgery, and repeat the above. To see that the pos. hom. class survives: attach N_n above to Ω; get something separated by hom. sphere $\Rightarrow$ homology splits $\Rightarrow$ pos. class survives. Also, $b_+^2(N_n) = 1$, hence it increases b_+^2. $\square$

5) Recall: $SW: \mathrm{spin}^c(M^4) \to \mathbb{Z}$: count solutions to SW equations.
Call $c \in H^2(M)$ basic if $\exists\, S \in \mathrm{spin}^c$ st $SW(S) \neq 0$ and $c_1(\mathcal{L}) = c$.
If $b_+^2 > 1$, $\mathrm{Basic}(M)$ is a diffeo invariant of M.
We saw earlier: if M symplectic, then $\pm c_1(M) \in \mathrm{Basic}(M)$ (we saw that SW are ± 1). There could be others, but:

Fact: if M is compact Kähler minimal of general type, then $\pm c_1(M)$ is the only basic class.

Ingredients to prove this: let $K = \Lambda^2 T^*M = -c_1(M)$

(1) minimal + general type $\Rightarrow K \cdot K > 0$ and $K \cdot [\omega] > 0$ (1)

(2) assume $c \in \text{Basic}(M)$; then $\dim(\mathcal{M}_c) \geq 0 = \dim(\mathcal{M}_\omega)$ ← from before: have a-$\mathbb{C}$ mfld

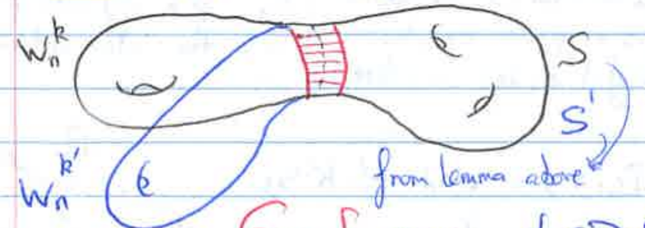
$\dim(\mathcal{M}_c) = c_1^2 - 2\chi - 3\sigma$ (since $SW(c) \neq 0$), $\quad \dim(\mathcal{M}_\omega) = K^2 - 2\chi - 3\sigma$

$\Rightarrow c_1^2 \geq K^2$ (2)

(3) Kähler $\Rightarrow$ Hodge theory: hd Δ and Riem Δ are the same. The Hodge index theorem can be used to play (1) and (2) against each other and derive a contradiction, unless $c = \pm K$.

6) **Theorem:** [Lisca–Matić] $\xi_n^k \underset{\text{iso}}{\simeq} \xi_n^{k'} \iff k = k'$ or $k = n - k'$.

Proof: $W_n^k \subseteq S$ as in the lemma



Because isom. cont. structures have isom. collars, so we can glue both to S.

Gompf: every $\phi \in \text{Diff}(\partial N_n)$ extends to N_n and acts as ± 1 on H^*.

This uses: (1) isotopy $\leftrightarrow$ homotopy for diffs (3D)
(2) do algebraic topology to classify diffeos.

So, id: $S \setminus W_n^k \to S' \setminus W_n^{k'}$ extends to a diffeo between S and S'. By diffeo invariance, $\phi^* \text{Basic}(S') = \text{Basic}(S)$

by fact above: $\phi^*\{\pm c_1(S')\} = \{\pm c_1(S)\}$

So $\phi^* c_1(W_n^k) = \pm c_1(W_n^{k'}) = \pm(2k' - n)\,PD(T)$

by Gompf: $\phi^* c_1(W_n^k) = \pm c_1(W_n^k) = \pm(2k-n)\,PD(T)$

Other direction: by the form of the surgery: flip orientation on S.

So $k = k'$ or $n - k'$. $\square$

Steven: Fact above $\Rightarrow$ c_1(Kähler str) is an ~~invariant up to~~ diffeomorphism invariant of the manifold.