23-May-2017

FILLINGS OF UNIT COTANGENT BUNDLES. TOM

There are so many strong fillings of unit cotangent bundles
Is there a unique exact filling of unit cotangent bundle?
there is at least one, the unit disk (
but are there others? unique for $g=0,1$ but not known for $g \geq 2$.

THEOREM 2: If: (Y_g, ξ_g) is the unit cotangent bundle of Σ_g, then the homology of any exact filling is that of the unit disk.

THEOREM 1: If (Y, ξ) admits some Calabi Yau ~~cap~~ ξ the order of the set of triples $b_1(N)$, $b_2(N)$ & $b_3(N)$ is finite where N is an exact filling.

THM 3: There is a unique Stein filling of (Y_g, ξ_g) up to s-cobordism boundary.
(simple cobordism)

simple homotopy equivalence = "cant crush things too crazy."

Jacob Lawry ? | 1-2 Calabi-Yau caps, uniruled caps & symplectic fillings by Lee, Mak, Yasui (LMY)

3 proven by (SV)

DEFn: A Calabi-Yau cap of a contact manifold is a compact (P, ω) which is a strong concave filling with $c_1(P)$ torsion

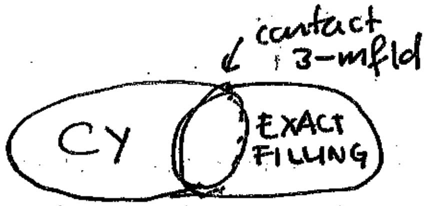
cap = concave filling

PROPOSITION/EXERCISE: The pairing is well defined

$$H^2(X; \mathbb{R}) \times H^2(X, \partial X; \mathbb{R})$$

$$[A] \cdot [(B,b)] \longrightarrow \int_X A \wedge B - \int_{\partial X} A \wedge B$$

Proof of theorem 1: Pick a Calabi Yau cap with Liouville contact form near boundary, and let (N, ω_N) be an exact filling

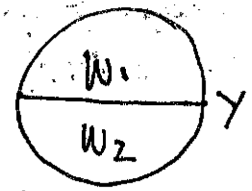


Scaling, or Liouville contact form and symplectic forms, we can take

$$c_1(W)\cdot[(\omega_N, \alpha_N)] = 0$$

LEMMA: If (W, ω_i) is an exact filling of (Y, ξ) with ~~Exact~~ Liouville 1-form ω & (Y,ξ) has a symplectic cap W_1 & exact filling W_2.
$\Rightarrow$ for sufficiently large $t > 0$ $\exists$ a symplectic form, ω on the glued mfld, with

$$c_1(X)\cdot W = c_1(W_1)\cdot[(\omega_1, \alpha_1)] + t\, c_1^{(W_2)}[(\omega_2, \alpha_2)]$$



$c_1(X)\cdot W = 0$ tells us about Kodaira dimensional $\rightarrow$ either ∞ $\rightarrow$ is minimal symp Calabi Yau.

THM
If (X, ω) is minimal symplectic Calabi-Yau, its rational homology is that of a K3 surface, the Enriques surface or a torus bundle over a torus.

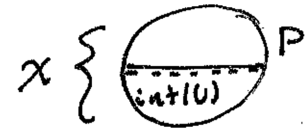
Let U be the unit cotangent disk bundle

[redacted]

You can use Kahler geometry
given an exact filling - get a lagrangian surface
g tori and a lag sphere

LEMMA 1: for each g There is a ~~surface~~ symplectic K3 surface X which containes g distinct lagrangian tori which represent the same homology class A and a lagrangian sphere which intersects each torus transversely at one point. ~~when X has the same homology of an exact filling~~. Call [redacted] the tori + sphere L
lag surgery.

Identify U with a tubular nhd of L



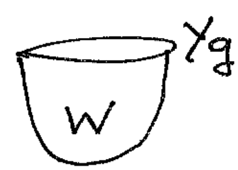
$P = X - \mathrm{int}(U)$ is a CY cap.

[redacted] look at intersection form of P & study its homology.
can only embed into some surfaces.
USE MVS + PD to get homology of [redacted] X to get proof of thm 2.

THM 3: ~~Reference~~: Fillings of Unit cotangent Bundles
Csivek, Van Horn-Morris) (SV)

THM If (W, J) is a stein filling of (Y_g, ξ_g)

$$\pi_1(W) \cong \pi_1(\Sigma_g).$$



$$\pi_1(Y_g) \cong \langle a_i, b_i, t \mid \prod [a_i, b_i] = t^{2g-2},\ [a_i, t] = [b_i, t] = 1 \rangle$$

A stein filling only consists of handles w/ index $k \leq 2$.

(turn it upside down).

$$\Rightarrow \quad i_* : \pi_1(Y_g) \twoheadrightarrow \pi_1(W)$$

Let $H \trianglelefteq \pi_1(Y_g)$ where $H = \langle a_i, b_i \rangle$ $\quad a_i, b_i \in \pi_1(Y_g)$

CLAIM: $i_* : H \twoheadrightarrow \pi_1(W)$

shown w/ covering spaces.

$$[\pi_1(Y_g) ; H] =: \; = 2g-2 \in \mathbb{N}$$

$$[\pi_1(W) ; i_*(H)] =: K \in \mathbb{N}.$$

$$(K-1)(2-2g) \geq -1$$

computation w/ euler characteristic

$K > 1$ doesn't $\Rightarrow K = 1$.

$\pi | K = 1$ works but
π
$g \geq 1$

We are going to argue that

PROPOSITION $\exists$ SES from $1 \to \langle \tau \rangle \hookrightarrow \pi_1(W) \twoheadrightarrow \pi_1(\Sigma_g) \to 1$

~~There exists~~ surface groups are RFRS

$$\exists \; G_0 := \pi_1(\Sigma_g) \trianglerighteq G_1 \trianglerighteq G_2 \geq \dots \geq$$

all are normal

$\nexists$ $\bigcap G_i = \{1\}$ and G_i / G_{i-1} is cyclic.

NOW, cyclic groups correspond to cyclic covers

$\Rightarrow$ group theory stuff $\Rightarrow$

Lemma $H_2(K(\pi_1(W)) ; \mathbb{Z}) \simeq \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{2g}$

$n = 1$