Tom - Fillings of unit cotangent bundles

yes for $g=0$ or 1; unknown for $g\geq 2$.

There are ∞^{ly} many strong fillings of those. Is there a unique exact one? There is clearly one: the unit disk bundle.

Theorem 2: if (Y_g, ξ_g) is the unit cotangent bundle of Σ_g, then the homology of any exact filling is that of the unit disk cotangent bundle.

Theorem 1: if (Y, ξ) admits some Calabi-Yau cap, then
$|\{(b_1(X), b_2(X), b_3(X)) \mid X \text{ exact filling of } Y\}| < \infty$.

Theorem 3: there is a unique Stein filling of (Y_g, ξ_g), up to s-cobordism rel boundary.

Ref for 1 & 2: "Calabi-Yau caps, Uniruled caps and symplectic fillings" by Li, Max, Yasui (=LMY).

Definition: a Calabi-Yau cap of a contact 3-manifold is a compact (P, ω) which is a strong concave filling with $c_1(P)$ torsion.

Proposition: the pairing $H^2_{dR}(X;\mathbb{R}) \times H^2_{dR}(X, \partial X;\mathbb{R})$, $([A],[(B,b)]) \mapsto \int_X A\wedge B - \int_{\partial X} A \wedge b$ is well defined.

Proof of theorem 1: pick a Calabi-Yau cap with Liouville contact form near boundary, and let (N, ω_N) be an exact filling: 

Together, they form a closed 4-manifold.
Scaling our Liouville contact form, we can take $c_1(P)\cdot[(\omega_N, \alpha_N)] = 0$

and also sympl form

Lemma: [LMY] if (W_i, ω_i) are ~~Calabi-Yau caps~~ exact fillings of (Y, ξ) with Liouville 1-form α, then for $t \gg 0$, there is a symplectic form ω on the glued manifold with ~~$c_1(X)\cdot\omega = c_1(N)\cdot[(\omega_N,\alpha_N)] + tc_1(P)\cdot[(\omega_P,\alpha_P)]$~~

$$c_1(X)\cdot\omega = c_1(W_1)\cdot[(W_1,\alpha_1)] + t\, c_1(W_2)[(W_2,\alpha_2)]$$

(W_1, ω_1) is a symplectic cap, (W_2, ω_2) is an exact filling. (or opposite)

So for us: $c_1(X)\cdot\omega = t c_1(N)\cdot[(\omega_N,\alpha_N)] + c_1(P)\cdot[(\omega_P,\alpha_P)]$

Theorem: if (X, ω) is a minimal symplectic Calabi-Yau, its rational homology is that of a K3 surface, the Enriques surface, or a torus bundle over a torus.

· This concludes the proof of theorem 1, at least in some cases. $\square$

Proof of theorem 2: let $\mathcal{U}$ be the unit cotangent disk bundle

Lemma: there is a symplectic K3 surface X which contains g distinct Lagrangian tori (all in the same homology class, and don't intersect) and a Lagrangian sphere which intersects each torus transversely at one point, ~~where X has the same homology as an exact filling.~~

Let L be the union of the tori and the sphere, glued together using Lagrangian surgery. It turns out that L is a genus g surface. Identify $\mathcal{U}$ with a tubular neighbourhood of L.

Then, $P = X - \mathrm{int}(\mathcal{U})$ is a Calabi-Yau cap. Try to conclude. $\square$

Ref for 3: "Fillings of unit cotangent bundles" by Sivek and van Horn-Morris.

We will focus on the following subtheorem.

Theorem: if (W, J) is a Stein filling of (Y_g, ξ_g), then $\underset{G}{\pi_1(W)} \cong \pi_1(\Sigma_g)$.

Proof Fact: $\pi_1(Y_g) \simeq \langle a_i, b_i, t \mid \prod [a_i, b_i] = t^{2g-2}, [a_i, t] = [b_i, t] = 1 \rangle$;

this uses the LES for fibers.

$\Rightarrow i_*: \pi_1(Y_g) \twoheadrightarrow \pi_1(W)$, since Stein: only 0, 1 and 2-handles.

Let $H \trianglelefteq \pi_1(Y_g)$ where $H = \langle a_i, b_i \rangle$. Claim: $i_*: H \twoheadrightarrow \pi_1(W)$.

Proof: $[\pi_1(Y_g) : H] = \cancel{k \in \mathbb{N}}\ 2g-2$

So, $[\pi_1(W) : i_*(H)] =: k \in \mathbb{N}$, $k \leq 2g-2$

"$\Rightarrow$" $(k-1)(2-2g) \geq -1$ by covering spaces theory

$\Rightarrow k = 1$, as $2-2g < 0$ by assumption. So, $i_* H \twoheadrightarrow \pi_1(W)$. $\square$

Proposition: $\exists$ SES $1 \to \langle \overset{\text{as above}}{t} \rangle \hookrightarrow \pi_1(W) \twoheadrightarrow \pi_1(\Sigma_g) \to 1$.

Proof: surface groups are RFRS $\Rightarrow \exists\ G_0 := \pi_1(\Sigma_g) \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots$, with quotients cyclic, and $\bigcap G_i = \{id\}$. Then use LHSerre SS and group cohomology, and Lemma: $H_2(\pi_1(W); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{2g}$.

So $n = 1$, so $\langle t \rangle = \{id\}$. $\square$