Jie

Symplectic Kodaira dim 0 (Actually, just some characterization of $\mathbb{Q}H_*(K3)$.) REFERENCE (Chapter 1 of Gompf+Stipschitz on 4 mflds)

Main THM symplectic 4-mfld, with $c_1 = 0$ $b_1 = 0$ $b_2^+ > 1$
$\Rightarrow$ its a $\mathbb{Q}H_*(K3)$.

(still true if c_1 torsion).

(taubes)
THM A X symplectic w/ $b_2^+ > 1$ then $SW_X(K_\omega) = \pm 1$

THM B (Morgan - Szabo) If X spin with $b_1 = \square$ $b_2^+ = 4n-1$
$b_2^- = 20n-1$, $n > 1$, then $SW(\square) \equiv 0 \mod 2$.

$$C_1 = C_1(TM) = C_1(\Lambda^{2,0} TM) = C_1(K_\omega)$$
canonical line bundle.

in thm B, determinant line bundle $\det(S_+) = $ trivial $\Rightarrow C_1 = 0$

spin$^{\mathbb{C}}$ structure on $X \longleftrightarrow H^2(X;\mathbb{Z})$

Proof of main thm assuming thm A+B.

$$b_1 = 0 \Rightarrow \chi = b_2^+ + b_2^- + 2$$

$$c_1(K_\omega) = 0 \Rightarrow c_1^2(K_\omega) = 3\sigma + 2\chi. \quad \text{Hirzebruch-signature thm.}$$

$$C_1(TM) = 0 \Rightarrow w_2(TM) = C_1(TM) \mod 2.$$

$\Rightarrow$ ~~[illegible]~~ M is spin

Rochlins thm $\Rightarrow 16 \mid \sigma$.

$$\sigma = b_2^+ - b_2^- = 16n$$

$$\Rightarrow b_2^+ = 4n-1 \quad b_2^- = 20n-1$$

trivial line bundle

By thm B if $n > 1 \Rightarrow SW(0) = 0 \pmod 2$

Symplectic $SW(K_\omega) =$ ~~[illegible]~~
$$\shortparallel$$
$$\pm 1$$

BUT $SW(0) = SW(K_\omega)$ $\lightning$

$\Rightarrow n = 1$

$\Rightarrow b_1 = 0 \quad b_2^+ = 3 \quad b_2^- = 19$ betti # of K3.

b/c M is spin $\Rightarrow Q_M$ is even

$\Rightarrow Q_M = 2E_8 \oplus 3H$ by classification of intersection forms

classification of unimodular nondegenerate quadratic forms Q (indefinite)

Q even if $Q(a,a)$ even for all $a \in H^2(X;\mathbb{Z})$

$\downarrow$

$Q = mE_8 \oplus nH$

$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Q odd otherwise

$\downarrow$

$Q = b_2^+\langle 1\rangle \oplus b_2^- \langle -1\rangle$

Proof Now we want to start proving THM A: //

Canonical Spin$^{\mathbb{C}}$-structure on (X, ω)

choose ω-compatible acs J, & ω is self dual wrt to some metric g.

$$\Lambda^{0,*}T^*X \longleftarrow \text{Spin}^{\mathbb{C}} \text{ bundle}$$

$$\begin{array}{c} \| \\ (\Lambda^{0,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{0,1} \\ \| \qquad\qquad \| \\ S_+ \qquad\qquad S_- \end{array}$$

Clifford multiplication $T^*X \otimes \mathbb{C}$ on [redacted] $\Lambda^{0,*}(T^*X)$ is given by $cl(v)\cdot\alpha = \sqrt{2}\,(v^{0,1}\wedge\alpha - i(\overline{v^{1,0}})\,\alpha)$

$v \in \Gamma(T^*X \otimes \mathbb{C}) \quad \alpha \in \Gamma(\Lambda^{0,*}(T^*X))$

$S_+ = \underline{\mathbb{C}} \oplus K^{-1}$

$S_- = TX$

~~U(2) ⊗ SO(4)~~

$$\begin{tikzcd} & & SU(2)\times SU(2)\times S^1 \arrow[d, phantom, "\|"] & \\ SU(2)\times S^1 = U(2) \arrow[rr, hook] & & \text{Spin}^{\mathbb{C}}(4) \arrow[dl, "S_+"'] \arrow[dr, "S_-"] & \\ & U(2) & & U(2) \end{tikzcd}$$

$U(2) \longrightarrow \text{Spin}^{\mathbb{C}}(4)$

$(A, \lambda) \longmapsto (\lambda, A, \lambda)$.

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ \begin{pmatrix} 1 & \\ & \det A \end{pmatrix} & & A \end{tikzcd}$$

$\Psi \in \Gamma(S_+)$ can be written as (α, β) where $\alpha \in \Gamma(\underline{\mathbb{C}}) = C^\infty(X)$, $\beta \in \Gamma(K^{-1})$.

$$\{(D_A\Psi = 0 \quad F_A^+ = q(\Psi) + i\mu)\}$$

$$F_A^+ = i\underbrace{(|\alpha|^2 - |\beta|^2)\omega + 2(\bar\alpha\beta - \alpha\bar\beta)}_{q(\Psi)} + i\mu.$$

In this cases dim $(\mathcal{M}) = 2d = 0$.
$= \frac{1}{4}(c_1(K_\omega) - (3\sigma + 2\chi)) = 0$

$u_0 = 1 \in \Gamma(S_+)$

$\exists$ spin$^{\mathbb{C}}$ connection A_0 s.t. $\nabla_{A_0} u_0 \in \Omega^1(X, K^{-1})$

Proof: Choose any connection A

$A + a \quad a \in \Omega^1(i\mathbb{R})$

$\nabla_{A+a} u_0 = \nabla_A u_0 + \frac{1}{2} a u_0$

$a = (-2\nabla_A u_0)_{\mathbb{C}} \qquad A_0 = A + a$

With $A_0 \rightsquigarrow D_{A_0} u_0 = 0$

Missed: $S_+ = \underline{\mathbb{C}} \oplus K^{-1}$

$cl(\omega)$ acts on S_+ bundle $\longrightarrow$ get eigenspaces
symplectic form

$S_+ = \underset{-2i}{\mathbb{C}} \oplus \underset{2i}{K^{-1}}$

KEY LEMMA: $D_{A+2a}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \bar\partial_a \alpha + \bar\partial_a^* \beta$

$\bar\partial_a = (\nabla_a)^{0,1}$

Now we can rewrite the SW equ into:

$\bar\partial_a \alpha = -\bar\partial_a^* \beta$

$F_A^+ = i(|\alpha|^2 - |\beta|^2)\omega + i(\bar\alpha\beta - \alpha\bar\beta) + i\mu$

Strategy: use a specific perturbation term $\mu = -r\omega - iF_{A_0}^+$ where $r \in \mathbb{R}_+$
~~So now we have two eqns~~

~~$\langle F_A^+, \omega\rangle = i(|\alpha|^2 - |\beta|^2)$~~

~~$F_A^{0,2} = i\bar\alpha\beta$~~

$A = A_0 - a$

$(da)^{0,2} = r\bar{\alpha}\beta \qquad r \in \mathbb{R}_+$

$\langle \omega, da \rangle = ir(1 - |\alpha|^2 + |\beta|^2)$

Now, we have that $\qquad \alpha \in \Gamma(L)$ here $L = \mathbb{C}$

$\Psi = \sqrt{r}\,(\alpha, \beta)$

$\Rightarrow \int \left(\left(1 - \frac{2c}{r}\right) |\nabla_a \alpha|^2 + r(1 - |\alpha|^2)^2 \right) \leq \underbrace{2\pi[\omega]\cdot c_1(\mathbb{C})}_{\text{Chern-Weil theory}} \leq 0$

$\Rightarrow$

$\bar\partial_a \bar\partial_a^* \beta = -\bar\partial_a \bar\partial_a \alpha$
$= -F_a^{0,2} + \alpha/(\nabla_a \alpha).$

$\Rightarrow$ for large enough r

- $\nabla_a \alpha = 0$ & $|\alpha| = 1$

SOLN: $\alpha \cong 1$, $\beta = 0$, $a = 0$ b/c α is const.

$\approx$ Gauge group

$\Rightarrow$ theres only one soln

$\Rightarrow SW_X(k\omega) = \pm 1.$

Proof of thm B

Uses: Pin(2) action on $\mathcal{M}$

subgroup of $Sp(1)$ generated by j, S^1 &

$j \circlearrowright \mathbb{C}^2 \quad (z_1, z_2) \mapsto (-\bar{z}_2, \bar{z}_1)$

So, $Pin(2) \circlearrowright \mathcal{M}_0$ where $\mathcal{M} = \mathcal{M}_0 / S^1$

$$\mathcal{F} : \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-)$$

$$\mathcal{F}\begin{pmatrix} a \\ \psi \end{pmatrix} = \begin{pmatrix} D_{A_0+a}\psi \\ F^{-1}_{A_0+a} - q(\psi) + i\mu \end{pmatrix}$$

$\Rightarrow$ $(A_0, \psi = 0)$ is reducible soln and away from this soln, action becomes an involution fixed pt free on $\mathcal{M} - [A_0, 0]$

$\Rightarrow$ $\mathcal{M} - [A_0, 0] \rightsquigarrow$ even SW

$[A_0, 0]$ locally modeled by

$$Q : \mathbb{H}^n \to \mathbb{R}^{4n-1}$$

Fredholm operator mod out $\mathcal{M}$ & look at kernel + cokernel which correspond to $\mathbb{H}^n$ & $\mathbb{R}^{4n-1}$ here.

Let

$$L = d\mathcal{F}_{[A_0, 0]} : \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-)$$

$$\begin{array}{ccc} Q : \ker(L) & \longrightarrow & \operatorname{Coker}(L) \\ \| & & \| \\ \ker(D) & \longrightarrow & \operatorname{Coker} D \oplus H^2_+ \end{array}$$

D = dirac operator $= D_{A_0}$

Locally around $[A_0, 0]$ $\mathcal{M}$ is $Q^{-1}(0)/S^1$.

$\Rightarrow$ $P : \ker(D) \to \operatorname{Coker}(D)$

$\Rightarrow$ $Q|_{\ker(P)} : \mathbb{H}^n \to \mathbb{R}^{4n-1}$

UPSHOT $(n = 1)$

$\Rightarrow$ $\dim \ker = 1$

$\Rightarrow$ $\ker / S^1 = \pm 1 = SW$

$(n > 1)$

$\dim \ker > 1$

$\Rightarrow$ SW <u>even.</u>