Jie - Symplectic Kodaira dimension 0

Actually: just some characterization of $\mathbb{Q}H_*(K3)$

Main theorem: symplectic 4-mfld, with $c_1=0$, $b_1=0$, $b_2^+>1$.
Then, it's a $\mathbb{Q}H_*(K3)$.
Rem: still true if c_1 torsion.

Theorem A: [Taubes] X symplectic with $b_2^+>1$, then $SW_X(K_\omega)=\pm 1$.
Theorem B: [Morgan-Szabo] if X spin with $b_1=0$, $b_2^+=4n-1$, $b_2^-=20n-1$, $n>1$, then $SW(0)=0 \mod 2$.
↳ trivial spin$^{\mathbb{C}}$ structure.

Here, $K_\omega=\Lambda^{2,0}TM \rightsquigarrow c_1=c_1(TM)=c_1(\Lambda^{2,0}TM)=c_1(K_\omega)$, for ω a symplectic form.
0 is the trivial spin$^{\mathbb{C}}$-structure: $\det(S_+)=$ trivial, $c_1=0$.
Spin$^{\mathbb{C}}$ on $X \longleftrightarrow H^2(X;\mathbb{Z})$: every 2 spin$^{\mathbb{C}}$-structures differ by tensoring a line-bundle, and that difference is c_1.

Proof of main theorem
$b_1=0 \Rightarrow \chi=b_2^++b_2^-+2$ by Poincaré duality.
$c_1^2(K_\omega)=3\sigma+2\chi$ by Hirzebruch signature theorem,
so $c_1(K_\omega)=0 \Rightarrow 3\sigma+2\chi=0$.
Also, $c_1(TM)=0 \Rightarrow w_2(TM)=c_1(TM) \mod 2=0$, so M is spin ($w_2=$ obstruction to being spin).
By Rochlin's theorem: $16|\sigma$, so $\sigma=b_2^+-b_2^-=16n$.
$\Rightarrow b_2^+=4n-1$, $b_2^-=20n-1$.

By theorem B, if $n>1$, then $SW(0)=0 \mod 2$. But the manifold is symplectic, so by theorem A, $SW(K_\omega)=\pm 1$.
So we have $n=1$, otherwise there is a contradiction.
$\Rightarrow b_1=0$, $b_2^+=3$, $b_2^-=19$.

M spin $\Rightarrow$ intersection form Q_M is even $\Rightarrow Q_M=2E_8\oplus 3H$,
so we have a $\mathbb{Q}H_*(K3)$. □
Rem: for background, see chapter 1 of Gompf-Stipsicz.

Recall classification of intersection forms: we say Q is even if $Q(a,a) \equiv 0 \mod 2$ for all $a \in H^2(X;\mathbb{Z})$. ~~For Q~~ We have

$$Q \begin{cases} \text{even: } mE_8 \oplus nH, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \text{odd: } b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle \end{cases}, \quad \text{for } Q \text{ indefinite.}$$

m and n are linear functions of Betti numbers: E_8 are all negative, and H has 1 pos and 1 neg.

Proof of theorem A:

We need to know what the canonical spin$^{\mathbb{C}}$-structure on (X,ω) is. (Choose J s.t. ω is self-dual). $\Lambda^{0,*}T^*X$ is spin$^{\mathbb{C}}$-bundle, and it splits as $\Lambda^{0,*}T^*X = \underbrace{(\Lambda^{0,0} \oplus \Lambda^{0,2})}_{=S_+} \oplus \underbrace{\Lambda^{0,1}}_{=S_-}$.

The Clifford multiplication of $T^*X \otimes \mathbb{C}$ on ~~S_+~~ $\Lambda^{0,*}T^*X$ is given by
[pick g such that ω is self-dual, and then take the associated J]

$$cl(v).\alpha = \sqrt{2}\left(v^{0,1} \wedge \alpha - \iota(\overline{v^{1,0}})\alpha\right) \qquad v \in T^*X \otimes \mathbb{C},\ \alpha \in \Lambda^{0,*}T^*X$$

We have $S_+ = \underline{\mathbb{C}} \oplus K^{-1}$, and $S_- = TX$. For S_+, this is the splitting of $cl(\omega)$ into $(-2i)$-eigenspace and $(2i)$-eigenspace.

There is a canonical inclusion $U(2) \hookrightarrow \mathrm{Spin}^{\mathbb{C}}(4)$, for A a transition matrix for the manifold

$$\begin{array}{ccc} & \mathrm{Spin}^{\mathbb{C}}(4) & \\ {\scriptstyle S_+} \swarrow & & \searrow {\scriptstyle S_-} \\ U(2) & & U(2) \\ \begin{pmatrix} 1 & \\ & \det A \end{pmatrix} & \longleftarrow A \longrightarrow & A \end{array}$$

Any $\psi \in \Gamma(S_+)$ is (α,β) where $\alpha \in \Gamma(\underline{\mathbb{C}}) = C^\infty(X)$, $\beta \in \Gamma(K^{-1})$.
Now, the $q(\psi)$ term in the SW-equations has a better form: the equation becomes
$$F_A^+ = \underbrace{i(|\alpha|^2 - |\beta|^2)\omega + 2(\bar{\alpha}\beta - \alpha\bar{\beta})}_{q(\psi)} + i\mu$$

And in this case, $\dim \mathcal{M} = 2d = 0 = \frac{1}{4}\left(c_1^2(K_\omega) - (3\sigma + 2\chi)\right) = 0$

Take $u_0 = 1 \in \Gamma(S_+)$; $\exists$ spin$^{\mathbb{C}}$ connexion A_0 such that $\nabla_{A_0} u_0 \in \Omega^1(X, K^{-1})$. Indeed: choose any spin$^{\mathbb{C}}$ connexion A; any other will be $A + a$ for $a \in \Omega^1(i\mathbb{R}) = \mathrm{Lie}\, U(1)$; we have
$\nabla_{A+a} u_0 = \nabla_A u_0 + \frac{1}{2} a u_0$ ← Clifford multiplication. Take $a = (-2\nabla_A u_0)_{\underline{\mathbb{C}}}$ (projection),
and $A_0 = A + a$: this way, $\nabla_{A_0} u_0$ sits in $\Omega^1(X, K^{-1})$.
And we have: $A_0 \rightsquigarrow D_{A_0} u_0 = 0$.

Lemma: $D_{A_0+2a}\binom{\alpha}{\beta} = \bar\partial_a \alpha + \bar\partial_a^* \beta$, where $\bar\partial_a = (\nabla_a)^{0,1}$.

With this, the SW-equations become $\begin{cases} \bar\partial_a \alpha = -\bar\partial_a^* \beta \\ F_A^+ = i(|\alpha|^2 - |\beta|^2)\omega + 2(\bar\alpha\beta - \alpha\bar\beta) + i\mu \end{cases}$.

Now, we will use a specific perturbation term: take $\mu = -r\omega - iF_{A_0}^+$, where $r \in \mathbb{R}_+$. Then, the equations become $\begin{cases} (da)^{0,2} = r\bar\alpha\beta \\ \langle \omega, da\rangle = ir(1 - |\alpha|^2 + |\beta|^2) \end{cases}$, where $A = A_0 + a$.

Rescaling $\psi = \sqrt{r}(\alpha, \beta)$, we get the inequality

$$\int \left(\left(1 - \frac{2c}{r}\right) |\nabla_a \alpha|^2 + r(1 - |\alpha|^2)^2 \right) \leq \underbrace{2\pi [\omega] \cdot c_1}_{\text{Chern-Weil theory}} \quad (\mathfrak{E}) = 0$$

Taking r large enough: need $|\alpha| = 1$, $\nabla_a \alpha = 0$.

So $\alpha \equiv 1$, $\beta \equiv 0$, $a = 0$ is the unique solution, up to gauge transformation. This proves the Theorem A: for any other choice of μ, get something cobordant to 1 point $\Rightarrow$ sum of points is ± 1. $\square$

Proof of theorem B:

This part uses the $Pin(2)$ action on $\mathcal{M}$, where $Pin(2)$ is the subgroup of $Sp(1)$ generated by j and S^1, where $j \curvearrowright \mathbb{C}^2$ by $(z_1, z_2) \mapsto (-\bar z_2, z_1)$ (i.e. quaternion multiplication on the right).

Recall $\mathcal{F}: \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-) : (a, \psi) \mapsto \begin{cases} D_{A_0+a}\psi \\ F^+_{A_0+a} - q(\psi) + i\mu \end{cases}$

There is a $Pin(2) \curvearrowright \mathcal{M}_0$, and $\mathcal{M} = \mathcal{M}_0 / S^1$.

Since $Pin(2)$ is gen. by j and S^1, j acts on $\mathcal{M}$.

$(A_0, 0)$ is one reducible solution. Away from there, j is a fixed point-free involution, so the solutions in $\mathcal{M} - (A_0, 0)$ come in pairs.

Near $(A_0, 0)$, we have $L = d\mathcal{F}_{(A_0,0)} : \Omega^1 \oplus \Gamma(S_+) \to \Omega^2_+ \oplus \Gamma(S_-)$, and we have the local model $\underset{\text{"Kuranishi map"}}{Q} : \underset{\overset{\shortparallel}{\operatorname{Ker} D_{A_0}}}{\operatorname{Ker}(L)} \to \underset{\overset{\shortparallel}{\operatorname{Coker} D_{A_0} \oplus H_+^2}}{\operatorname{coker}(L)}$; for $P: \ker D_{A_0} \to \operatorname{coker} D_{A_0}$

$\operatorname{ind} D_{A_0} = -\frac{5}{4}$ uses b_2 b_2^+

get $Q|_{\ker P} : \mathbb{H}^{n} \to \mathbb{R}^{4n-1}$. For $n = 1$, $\dim \ker = 1$, so $\ker / S^1 = \pm 1$. But if $n > 1$, $\dim \operatorname{Ker} > 1$, so it gives many directions to deform the solutions. In any dir, can go in $+$-dir or $-$-dir, so they come in pairs, hence $SW(\mathfrak{s}) = 0 \bmod 2$. $\square$