INTRO TO SW (Reference: Hutchings–Taubes: An introduction to SW equs on symplectic mflds (§2-3))

SW Equs: on (A, Ψ) where A is a connection on a line bundle, Ψ spinnor, section of a complex $\mathbb{C}^2$ ~~bun~~ bundle.

$$D_A \Psi = 0$$

$$F_A^{+} = q(\Psi) + i\mu$$

$\Psi \in \Gamma(S_+)$

ON a 4-mfld

<u>Spin$^{\mathbb{C}}$-structures + associated bundles</u>

For $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$

$Spin(n) =$ connected double cover of $SO(n)$.

$Spin^{\mathbb{C}}(n) = (Spin(n) \times U(1))/\mathbb{Z}_2$

A spin$^{\mathbb{C}}$-structure on X

Given (X, g), you can form an $SO(n)$ bundle $\begin{matrix} Fr \\ \downarrow \\ X^n \end{matrix}$ s.t. fiber ~~at~~ $F_x =$ Isometries$(\mathbb{R}^n, T_x X)$

g Riemannian metric

Now

$$\begin{tikzcd} \hat{F} \times Spin^{\mathbb{C}}(n) \arrow[r] \arrow[d] & \hat{F} \arrow[d] \\ Fr \times SO(n) \arrow[r] & Fr \arrow[d] \\ & X \end{tikzcd}$$

THEOREM: X^4 oriented has a spin$^{\mathbb{C}}$-structure & furthermore its an affine space modelled on $H^2(X;\mathbb{Z})$

i.e every $\alpha \in H^2(X;\mathbb{Z})$ gives a spin$^{\mathbb{C}}$-structure.

Associated bundles:
Given a representation
$$\rho : G \to \mathrm{Aut}(V)$$

&
$$\begin{array}{ccc} G \to P & & \\ \downarrow & \Rightarrow & V \to P \times_\rho V \\ X & & \downarrow \\ & & X \end{array}$$

principal G bundle

Now we want to understand ~~rep~~ this when G is
spin$^{\mathbb{C}}$ ...?

- $\mathbb{C}^2$ bundles $S_\pm$ from: $\mathrm{Spin}^{\mathbb{C}}(4) \xrightarrow{S_\pm} U(2)$, $\mathrm{Spin}^{\mathbb{C}}(4) \xrightarrow{\kappa} U(1)$

(called the spinor bundles)

- Line bundle L from: ...

(called determinant line bundle) $\det S_\pm = \Lambda^2 S_\pm$

$$SO(4) \simeq SU(2) \times SU(2)/\pm 1$$

Proof: $\mathbb{R}^4 = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \ a, b \in \mathbb{C} \right\}$

$$(h_-, h_+) \in SU(2) \times SU(2)$$

$$(h_-, h_+) \cdot x = h_- x h_+^{-1}$$

$$\mathrm{Spin}(4) = SU(2) \times SU(2)$$

$$\Rightarrow \mathrm{Spin}^{\mathbb{C}}(4) = SU(2) \times SU(2) \times U(1)/\pm 1.$$

$$\ni$$

$$[h_-, h_+, \lambda]$$

$U(2) = SU(2) \times U(1)/\pm 1$

$K([h_-, h_+, \lambda]) = \lambda^2$

$S_+([h_-, h_+, \lambda]) = [h_+, \lambda]$

$S_-([h_-, h_+, \lambda]) = [h_-, \lambda]$

Clifford multiplication:

A map $cl : T^*X \longrightarrow \mathrm{End}(S_+ \oplus S_-)$

PROPERTIES

- $cl(v)$ interchanges S_+ & S_- for all v
- $cl(v)^2 = -|v|^2$
- $|v| = 1 \Rightarrow cl(v)$ unitary.

CONSTRUCTION:

$cl : T^*X \otimes S_+ \longrightarrow S_-$

at a point $\mathbb{R}^4 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$$x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \psi \longmapsto x\psi$$

$[h_-, h_+, \lambda] \in \mathrm{Spin}^{\mathbb{C}}(4)$

$h_- x h_+^{-1} \cdot \lambda h_+ \psi = \lambda h_- x \psi.$

$T^*X \otimes S_- \longrightarrow S_+$

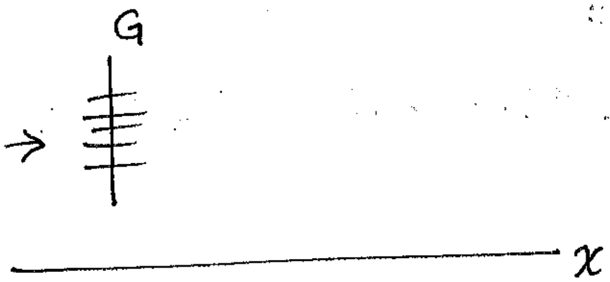
instead $(x, \psi) \longmapsto -\bar{x}^t \psi.$

~~[illegible]~~ Given spin$^{\mathbb{C}}(4)$ structure $\begin{matrix}\hat{F}\\ \downarrow \\ X\end{matrix}$

we want a connection.

In the case of a principal G-bundle, a connection is Lie(Spin$^{\mathbb{C}}(4)$)-valued-1-form on $\hat{F}$ satisfying same properties

want an G equivariant horizontal subspace



"Lie (Spin$^{\mathbb{C}}(4)$) tells you how to travel horizontally".

$$\text{Lie}(\text{Spin}^{\mathbb{C}}(4)) = \text{Lie}(SO(4)) \oplus \text{Lie}(U(1))$$

$\leadsto$ Levicivita lands in Lie(SO(4)) & a connection on L will land on Lie(U(1)) $\sim$.

connections as horizontal distributions or think of as

$\nabla_A : C^\infty(E) \to C^\infty(T^*X \otimes E)$ on a v.b. E

& its levi civita if its compatible w/ the metric & torsion free

$$dg(X,Y) = g(\nabla_A X, Y) + g(X, \nabla_A Y).$$

mnemonic: 'partial derivatives commute'.

Given a metric compatible connection A on L

set $D_A = C^\infty(S_\pm) \xrightarrow{\nabla_A} C^\infty(T^*X \otimes S_\pm) \xrightarrow{cl} C^\infty(S_\mp)$

Given $A \xleftrightarrow{\text{curvature}} F_A \in C^\infty(i\Lambda^2 T^*X)$

a metric gives you hodge $*$

$\rightsquigarrow$ can break up into eigenspaces $\Lambda^2 T^*X$

$$\Lambda^2 T^*X = \Lambda^2_+ T^*X \oplus \Lambda^2_- T^*X.$$

F_A^+

can extend clifford multiplication to

$$\Lambda^2 T^*X \longrightarrow \mathrm{End}(S_+)$$
$$v \wedge w \longmapsto \tfrac{1}{2}[cl(v), cl(w)]$$

In particular,

$$cl_+ : \Lambda^2_+ T^*X \otimes \mathbb{C} \longrightarrow \mathrm{End}(S_+)$$

$$q(\psi) = \overset{\text{dagger}}{cl_+^*}(\psi \otimes \psi^*).$$ $\leftarrow$ q from SW equs

$\mu \in C^\infty(\Lambda^2_+ T^*X)$.

So $\left\{ \begin{array}{l} D_A \psi = 0 \\ F_A^+ = q(\psi) + i\mu \end{array} \right\}$

depends on metric g, spin$^{\mathbb{C}}$-structure & μ.

" splitting ~~of~~ into S_+ & S_- & clifford mult = spin$^{\mathbb{C}}$-structure "

$\boxed{\text{WHAT IS THE INVARIANT?}}$

Idea: $\mathcal{F}(A, \psi) = \begin{pmatrix} D_A \psi \\ F_A^+ - q(\psi) - i\mu \end{pmatrix}$

want to study $\{\mathcal{F}(A,\psi) = 0\} = m_{g,A,\mu}$

moduli space

$\mathcal{M} = \mathfrak{m} / \mathcal{G}$ nice (ish) mfld $\rightsquigarrow$ extract homological invariants

if $\dim(\mathcal{M}) \neq 0 \Rightarrow$ don't have invariants.

<u>Gauge</u> <u>Group</u>: $\mathcal{G} = C^\infty(X, S^1)$ given by:

$u \cdot (A, \Psi) = (A - 2u^{-1}du, u\Psi)$.

this action is free except if $\Psi \equiv 0$

($\Psi = 0$ reducible case) $\Rightarrow$ get S^1-stabilizer

$\mathcal{M} = \mathfrak{m} / \mathcal{G}$ $\qquad \mathcal{M}^\circ = \mathfrak{m} / \mathcal{G}^\circ = \{ \phi \in \mathcal{G} \mid \phi(*) = 1 \}$

$\mathcal{M}^\circ$ might be smooth

$\mathcal{G}^\circ$ acting on $\mathfrak{m}$ is always free.

THEOREM: Fix f, g

(a) $\mathcal{M}$ is compact

(b) For generic μ, $\mathcal{M}^\circ$ is a smooth finite dimensional manifold w/ a smooth S^1 action.

If $b_{2,+} \geq 0 \Rightarrow$ for generic μ, $\mathcal{M}$ is also smooth and $\mathcal{M}^\circ \to \mathcal{M}$ is a principal S^1-bundle.

(c) For generic μ

$2d := \dim \mathcal{M} =$ topological quantity (only involving $X, c_1(L)$)

$=$ the thing that vanishes for a.c.s

$$2d = b_1 - 1 - b_{2,+} + \tfrac{1}{4}\big(c_1(L)\cdot c_1(L) - \underset{\substack{\| \\ b_{2,+} - b_{2,-}.}}{\tau} \big)$$

(d) orientations on $\mathcal{M}$ correspond to choosing vector space orientations on $H^0(X;\mathbb{R}) \oplus H^1 \oplus H^2_+$

(e) Generic path (g_t, μ_t) gives a smooth compact oriented cobordism on $\mathcal{M}^0$. Same on $\mathcal{M}$ if $b_{2,+} > 1$.

(b)–(e) std index / elliptic theory results

(once you realize that reducibles are codim $b_{2,+}$ in $\mathcal{M}$.)

$F_A{}^+ = q(\psi) + i\mu$ you have reducible solus when $q(\psi) = 0$

$\Downarrow$ harmonic $=$ $\in H_+^2$ $\Downarrow$ harmonic

~~[illegible]~~

~~[illegible]~~

DEFN

$SW_X : \mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{Z}$

- 0 if $b_{2,+} - b_1$ is even
- 0 if $d < 0$
- $d = 0 \Rightarrow SW_X(\mathfrak{s}) = \sum_{\mathcal{M}} \pm 1$
- $d > 0 \Rightarrow SW_X(\mathfrak{s}) = \int_{\mathcal{M}} e^d$ ~~[illegible]~~ $e \in H^2(\mathcal{M};\mathbb{Z})$.

THM:

$b_{2,+} > 1 \Rightarrow SW_X$ diffeomorphism invariant.

SW_X is finitely supported, (nonzero for fin many)

$SW_X \Longleftrightarrow SW_{X \# \overline{\mathbb{CP}^2}}$ contain the same information.

$X = Y \# Z \qquad b_{2,+}(Y, Z) > 0$

$SW_X = 0$

$\mathfrak{s} \to \bar{\mathfrak{s}} \quad \ldots\ldots \quad SW(\bar{\mathfrak{s}}) = \pm SW(\mathfrak{s})$

Bochner-Wietzenböck formula:

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{s}{4} + \frac{1}{2} cl(F_A)$$

s = scalar curvature

Lemma: If $\mathfrak{F}(A, \psi) = 0 \quad \psi \not\equiv 0$

$$|\psi|^2 \le \max_X \left(-\frac{s}{2} + 2|\mu|\right)$$

"for positive scalar curvature you don't get $SW_{\mathbb{Z}}$ invariance"

$\mathbb{C}P^2$ is the only 4-dim symplectic mfld with positive scalar curvature (uses a thm by Taubes)

which has $d = 0$

canonical spin^c-structure on symplectic mfld

$S_{\pm}$: Ω^{0}, $\Omega^{0,2}$

D ± const $= \bar{\partial} + \bar{\partial}^*$