Kevin - Introduction to Seiberg-Witten invariants

Ref: Hutchings-Taubes: An introduction to the SW equations on symplectic manifolds (§2-3).

SW-equations: $\begin{cases} D_A \Psi = 0 \\ F_A^+ = q(\Psi) + i\mu \end{cases}$, on a 4-manifold

Here, $(A, \Psi) \in \underset{\text{connection}}{\mathrm{Conn}(L)} \times \underset{\text{spinor}}{\Gamma(S_+)}$. And $\mu \in C^\infty(\Lambda^2_+ T^*X)$.

$\boxed{\triangle \text{ Equations in } (A,\Psi) \text{ depending on metric } g, \text{ spin}^{\mathbb{C}}\text{-structure and } \mu.}$

1) Spin$^{\mathbb{C}}$-structures and associated bundles:

For $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$, so let $Spin(n)$ be the connected double cover. Define also $Spin^{\mathbb{C}}(n) = (Spin(n) \times U(1))/\mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $Spin(n)$ by Deck transformations and on $U(1)$ by negation.

For $(X^n, g = \text{metric})$, let $\begin{matrix} Fr \\ \downarrow \\ X \end{matrix}$ such that the fiber over $x \in X$ is $\mathrm{Isom}(\mathbb{R}^n, T_xX)$; it is an $SO(n)$-bundle. Can we lift it to $Spin^{\mathbb{C}}(n)$? A lift is a $Spin^{\mathbb{C}}$-structure.

$$\begin{tikzcd} \hat{F} \times Spin^{\mathbb{C}}(n) \arrow[r] \arrow[d] & \hat{F} \arrow[d] \\ Fr \times SO(n) \arrow[r] & Fr \arrow[d] \\ & X \end{tikzcd}$$

Theorem: any X^4 oriented has a $spin^{\mathbb{C}}$-structure (an affine space modelled on $H^2(X;\mathbb{Z})$ worth of it, actually)

Associated bundles: for $\rho: G \to \mathrm{Aut}(V)$ and $\begin{matrix} G \to P \\ \downarrow \\ X \end{matrix}$, we can form the vector bundle $\begin{matrix} V \to P\times_\rho V \\ \downarrow \\ X \end{matrix}$.

Get 2 $\mathbb{C}^2$-bundles $S_\pm$ from $Spin^{\mathbb{C}}(4) \xrightarrow{S_\pm} U(2)$ and a line-bundle L from $Spin^{\mathbb{C}}(4) \xrightarrow{k} U(1)$, by taking $V = \mathbb{C}^2$ and $\mathbb{C}$ respectively.

We have $SO(4) \cong SU(2) \times SU(2)/\pm 1$. Indeed, write $\mathbb{R}^4 = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| \, a, b \in \mathbb{C} = \mathbb{R}^2 \right\}$, and $(h_-, h_+) \in SU(2) \times SU(2)$ acts by $(h_-, h_+).x = h_- x h_+^{-1}$.
So, also have $Spin(4) = SU(2) \times SU(2)$
and $Spin^{\mathbb{C}}(4) = SU(2) \times SU(2) \times U(1)/\pm 1 \ni [h_-, h_+, \lambda]$
Define $k([h_-, h_+, \lambda]) = \lambda^2$.

Also $U(2) = SU(2) \times U(1)/\pm 1$, and so we define $S_+([h_-, h_+, \lambda]) = [h_+, \lambda]$ and $S_-([h_-, \lambda])$.

Definition: $S_\pm$ are spinor bundles, and L is the determinant line bundle of S_+ (or S_-; they are equal): $\det S_\pm = \Lambda^2 S_\pm$.

Clifford multiplication: map $cl : T^*X \to End(S_+ \oplus S_-)$.
Properties: * $cl(v)$ interchanges S_+ and S_-, $\forall v$: $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.
* $cl(v)^2 = -|v|^2$
* $|v| = 1 \Rightarrow cl(v)$ is unitary.
Construction: want $cl : T^*X \otimes S_+ \to S_-$ (and same swapping S_+ and S_-).
At a point $\mathbb{R}^4 \times \mathbb{C}^2 \to \mathbb{C}^2 : (x, \psi) \mapsto x\psi$.

Suppose we have $x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, ψ, $[h_-, h_+, \lambda] \in Spin^{\mathbb{C}}(4)$; it acts on x as $h_- x h_+^{-1}$ and on ψ as $\lambda h_+ \psi$ and we have $h_- x h_+^{-1} \cdot \lambda h_+ \psi = \lambda h_- x \psi$, so we see that the representations match.
Also, define the other part of it: $T^*X \otimes S_- \to S_+ : (x, \psi) \mapsto -\bar{x}^t \psi$.

Given a $Spin^{\mathbb{C}}(4)$-structure $\hat{F} \to X$, we want a connection, ie a $Lie(Spin^{\mathbb{C}}(4))$-valued 1-form on $\hat{F}$ satisfying stuff. We have $Lie(Spin^{\mathbb{C}}(4)) = Lie(SO(4)) \oplus Lie(U(1))$ (since we took covers to define $Spin^{\mathbb{C}}(4)$).

Levi-Civita lands here, and a connection on L lands there.
Rem: for A on vector bundle E, get $\nabla_A : C^\infty(E) \to C^\infty(T^*X \otimes E)$.
We want "compatibility" with the metric: $d\,g(x,y) = g(\nabla_A x, y) + g(x, \nabla_A y)$, and ∇_A should be "torsion-free".

Definition: Given a metric-compatible connection A on L, let D_A to be the composition $C^\infty(S_\pm) \xrightarrow{\nabla_A} C^\infty(T^*X \otimes S_\pm) \xrightarrow{cl} C^\infty(S_\mp)$.
This explains the 1st of the SW-equations.

Also, A gives a curvature $F_A \in C^\infty(i\Lambda^2 T^*X)$. A metric gives a Hodge $*$, which in turns gives eigenspaces
$$\Lambda^2 T^*X = \underbrace{\Lambda^2_+ T^*X}_{} \oplus \Lambda^2_- T^*X\,;$$
Definition: F_A^+ is the part living here.

We extend $cl: \Lambda^2 T^*X \to End(S_\pm^{\ddagger}) : v \wedge w \mapsto \frac{1}{2}[cl(v), cl(w)]$.
In particular, restricting to the positive part:
$cl_+ : \Lambda^2_+ T^*X \otimes \mathbb{C} \to End(S_+)$, and we define
Definition: $q(\psi) = cl_+^*(\psi \otimes \psi^*)$.

Rem: on a symplectic manifold, there is a natural spin^c-structure where the SW-equations simplify to something easier.

2) What is the invariant?

Idea: $\mathcal{F}(A,\psi) = \begin{pmatrix} D_A\psi \\ F_A^+ - q(\psi) - i\mu \end{pmatrix}$; we want to study $m_{g,s,\mu} := \{\mathcal{F}(A,\psi) = 0\}$. Let $\mathcal{M} = m/\mathcal{G}$; it is a nice(-ish) manifold, from which we can extract homological invariants.
Rem: think of μ as a perturbation.

What is the gauge-group $\mathcal{G}$? Let $\mathcal{G} = C^\infty(X, S^1)$, and it acts as $u\cdot(A,\psi) = (A - 2u^{-1}du, u\psi)$. We can check that the SW-equations are invariant under that.
This action is free, except if $\psi = 0$ where we get a S^1-stabilizer: u being constant. Definition: (reducible)

So $\mathcal{M} = m/\mathcal{G}$, and let $\mathcal{M}^\circ = m/\mathcal{G}^\circ$, where $\mathcal{G}^\circ$ is: pick $* \in X$, and let $\mathcal{G}^\circ = \{\phi \in \mathcal{G} \mid \phi(*) = 1\}$. The idea is that this way, u can not be constant (unless it's $u \equiv 1$), so we don't have S^1-stabilizers as above $\Rightarrow$ free action.

So, $\mathcal{M}^\circ$ has more chances of being non-singular than $\mathcal{M}$.

Theorem: fix $\mathfrak{s}, g$.

(a) $\mathcal{M}$ is compact

(b) For generic μ, $\mathcal{M}^\circ$ is a smooth finite dimensional manifold with smooth S^1-action. If $b_{2,+} > 0$, then for generic μ, $\mathcal{M}$ is smooth and $\mathcal{M}^\circ \to \mathcal{M}$ is a principal S^1-bundle.

(c) For generic μ, $2d := \dim \mathcal{M}$ is a purely topological quantity (only involving X and $c_1(L)$), that vanishes for almost-$\mathbb{C}$-structures:

$$= b_1 - 1 - b_{2,+} + \frac{1}{4}\left(c_1(L)\cdot c_1(L) - \tau\right)$$

$$\tau = b_{2,+} - b_{2,-}.$$

$c_1^2 - 3\,\mathrm{sign} - 2\chi$: this is 0 if manifold is almost complex.

(d) Orientations on $\mathcal{M} \longleftrightarrow$ vector space orientations on $H^0(X;\mathbb{R}) \oplus H^1 \oplus H^2_+$.

(e) A generic path (g_t, μ_t) gives a smooth compact oriented cobordism on $\mathcal{M}^\circ$. Same on $\mathcal{M}$ if $b_{2,+} > 1$.

Rem: (b)-(e) are standard index elliptic theory results (once you realize that reducible solutions are codimension $b_{2,+}$ in μ). (ie there is a codim $b_{2,+}$ space of μ's that give reducible solutions = bad points)

Theorem: * $b_{2,+} > 1 \Rightarrow SW_X$ is a diffeomorphism invariant.

Definition: $SW_X : \mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{Z}$ is defined:

* 0 if $b_{2,+} - b_1$ is even
* 0 if $d < 0$
* $d = 0$: $SW_X(\mathfrak{s}) = \sum \pm 1$ (count $\mathcal{M}$)
* $d > 0$: $\int_{\mathcal{M}} e^d$, $e \in H^2(\mathcal{M};\mathbb{Z})$.

Rem: it's conjectured that this is 0 if $d > 0$. And proved for symplectic mflds.

Theorem continued: * SW_X is finitely supported

* $SW_X \longleftrightarrow SW_{X \# \overline{\mathbb{CP}^2}}$ contain the same information ($\exists$ blow up formula)
* $X = Y \# Z$, $b_{2,+}(Y,Z) > 0 \Rightarrow SW_X = 0$.
* $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$ has for effect $SW(\bar{\mathfrak{s}}) = \pm SW(\mathfrak{s})$.

Bochner-Wietzenböck formula: $D_A^* D_A = \nabla_A^* \nabla_A + \underbrace{s}_{\text{scalar curvature}}/4 + \frac{1}{2}\, cl(F_A)$.

Lemma: if $F(A,\psi) = 0$, $\psi \neq 0$, then $|\psi|^2 \leq \max_X \left(-s/2 + 2|\mu|\right)$.

$\Rightarrow$ if $s > 0$, for small μ, there is no solution.

This gives regularity for F $\rightsquigarrow$ Ascoli-Arzela $\rightsquigarrow$ compactness.