Sarah - Applications of Wendl's theorem to classification of fillings

Ref. Plamenevskaya - van Horn - Morris : Planar open books, Monodromy factorizations and Symplectic fillings.

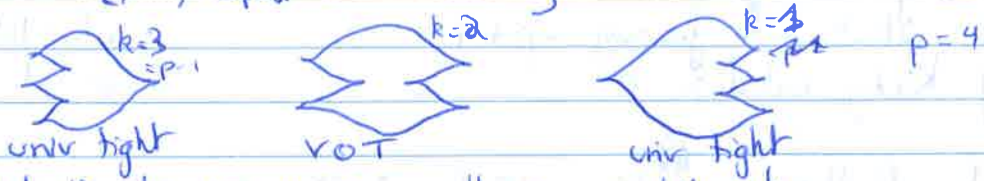
Recall: Stein fillings $\longleftrightarrow$ Positive factorizations of monodromies of comp. OB
Problem: need to know all open books compatible.
Theorem: if (Y, ξ) admits a planar OBD, then every strong symplectic filling is symplectically deformation equivalent to a blow-up of an allowable Lefschetz fibration, compatible with the given OB.

Definition: $L(p,1) :=$ quotient of $S^3 \subset \mathbb{C}^2$ by $\mathbb{Z}_p$-action $(z_1, z_2) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i}{p}} z_2)$.
Definition: (Y, ξ) is overtwisted if it contains an OT disk, otherwise it's tight.
If (Y, ξ) is tight and we lift it to the universal cover $(\tilde{Y}, \tilde{\xi})$ and it's tight again, then ξ is universally tight. Otherwise, it's ~~univ~~ virtually OT
(tight & not univ. tight)

OT / tight: univ tight, virt. OT

Goal: **Theorem**: every virtually OT contact structure on $L(p,1)$ has a unique Stein filling up to symplectic deformation, which is also its unique weak filling, up to symplectic deformation and blow up.
Corollary: if $p \neq 4$, then can replace "VOT" with "tight".

[Honda 2000] The tight contact structures on $L(p,1)$ are $\xi_1, \dots, \xi_{p-1}$, where ξ_k arises from Leg. surgery on the unknot with $p-2$ stabilizations: (stab = adding handle and Dehn twist) ~~we have~~ with k cusps on the left for pos. stab
The $\xi_2, \dots, \xi_{p-2}$ are virtually overtwisted. ⇁Z pos, ⇁S neg.



Total # of cusps is p. # cusps on left is k.

In the OB from Leg. surgery for ξ_k, the page is a disk with $n = p-1$ holes, and if δ_j = curve around j^{th} hole, the monodromy is
$\Phi = D_\alpha D_{\delta_1} \cdots D_{\delta_{k-1}} D_{\delta_{k+1}} \cdots D_{\delta_n} D_\beta$, where α contains the holes 1 through k, and β contains holes k through n.
$\Rightarrow$ Stein filling: $D^4 \cup$ 2-handles

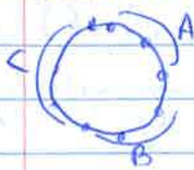
Rem. there is a procedure for going to OB to surgery diagrams, so, here, we can see these holes as the stabilizations:

see lemma 2 ← $D_\alpha D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n}$ corresponds to S^3, and adding the last D_β corresponds to doing the surgery.

here we use $k \neq p-1$, $k \neq 1$ and VOT. So need $p \geq 4$. Also, goes bad when $p \leq 4$.

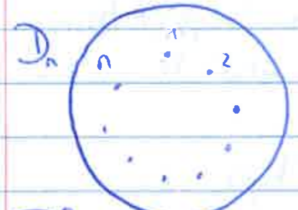
Lemma 1: any positive factorization of Φ takes the form $D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n} D_{\beta'}$, where α' and β' enclose the same loops as α and β.

Margalit/McCammond 2009: if D_n is the disk with n holes, $MCG(D_n)$ has a presentation with generators all convex Dehn twists and relations (i) Dehn twist along disjoint curves commute



(ii) Lantern relations: $D_A D_B D_C D_{A \cup B \cup C} = D_{A \cup B} D_{A \cup C} D_{B \cup C}$ if A, B, C are ~~orient~~ ordered clockwise.

⚠ maybe $A \cup B \cup C \neq$ all the holes



γ a simple closed curve is convex if it is isotopic to the boundary of the convex hull of some set of holes.

If γ is convex, D_γ factors into a product of ∂ and pairwise Dehn twists, using the lantern relations

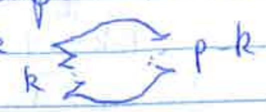
around a hole: δ_i around 2 holes:

Lemma 2: if $\Psi = D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n}$, then the OB with monodromy Ψ represents (S^3, ξ_{std}). The knot in S^3 induced by β' is the unknot with framing $-p+1$

ie include the page D_n inside S^3, and that's the knot

Proof: Kirby calculus. □

Proof of theorem: ~~Rem~~ X is LF for $D_{\alpha'} D_{\delta_1} \dots D_{\delta_{k-1}} D_{\delta_{k+1}} \dots D_{\delta_n} D_{\beta'}$; then by Lemma 2, it is diffeomorphic to D^4 with a 2-handle attached along the unknot coming from β' with framing $-p+1$. X has a Stein structure, coming from surgery on a knot with $tb = -p+1$. It's also (apparently) easy to compute the rotation number, so we know the knot, and it turns out to be



This is the only way to get $(L(p,1), \xi_k)$. The compatible symplectic structure is unique up to symplectic deformation, by Gompf 2004.
$\Rightarrow$ All Stein structures are the same.
$\Rightarrow$ Stein fillings are unique, by Wendl: we don't have to use any other OBD; it's enough to do it with <u>this one</u>.

<u>So</u>: by Wendl's theorem, we know there is a unique strong filling (which happens here to be Stein).

To prove the statement about weak fillings in the theorem, we invoke the following theorem:

Theorem [Ohta-Ono 1999] any weak filling of a rational homology sphere can be turned into a strong filling.

$\square$