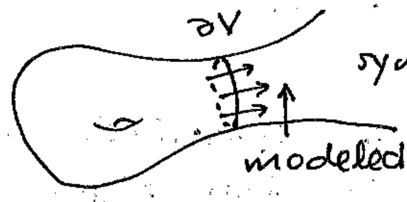
UMUT:

$(M^{2n+1}, \xi)$ compact connected mfld

$(V^{2n}, \omega, \alpha)$ Liouville mfld, suppose it is finite type

( i.e V has an end



)

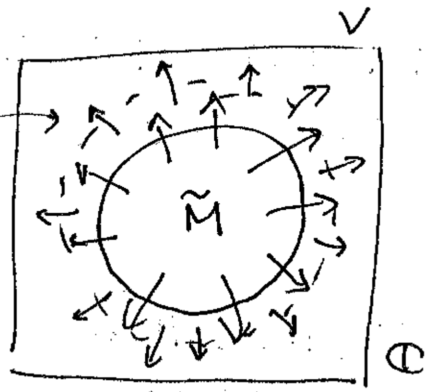
where $\alpha$ = Liouville vector field.

DEFINITION: M is V-spliffable if M is contactomorphic to a convex hypersurface $\tilde{M} \subset V \times \mathbb{C}$ such that $\tilde{M}$ divides $V \times \mathbb{C}$ into two subsets one of which is modeled after the positive symplectization of $\tilde{M}$

( $\mathbb{C}$ with std radial Liouville structure )

positive symp of M

in this component $\alpha$ is never 0




and

$\alpha \pitchfork \varphi_\alpha^t(\tilde{M})$ for $t \geq 0$

and

$\bigcup_{t \geq 0} \varphi_\alpha^t(\tilde{M})$ covers it.

THEOREM: (Berth-Geiges-Zehmish '16) M is V-spliffable

Let W be a spherical strong filling then $\exists$ diagrams of the following form

$$\begin{tikzcd} H_*(V) \arrow[r, two heads] \arrow[dr] & H_*(W) \\ & H_*(M) \arrow[u] \end{tikzcd}$$

Moreover
$$\begin{tikzcd} \pi_1(V) \arrow[rr, two heads] \arrow[dr] & & \pi_1(W) \\ & \pi_1(M) \arrow[ur] & \end{tikzcd}$$

REMARK 1: For the surjection involving $M$ & $W$ only we don't need to know what $V$ is.

THEOREM (Cielibach, Cielibach-Eliashberg)
If $M$ admits a subcritical Weinstein filling, then it is spliffable
$\downarrow$
(handles have index $\leq$ half dim of handlebody)

REMARK: (BGZ) prove a lot stronger statement about topological classification of filling

COROLLARY 1: (Eliashberg-Floer-McDuff) Let $W$ be a spherical filling of $(S^{2n-1}, \xi_{std})$ and assume $n \geq 3$. Then $W$ is diffeomorphic to $\mathbb{D}^{2n}$.

Proof: $S^{2n-1}$ is $\mathbb{C}^{n-1}$-spliffable. Here $V = \mathbb{C}^{n-1}$ + smooth topology. (h-cobordism thm).

Note: Proof of BGZ is a careful generalization of Eliashberg-Floer-McDuff proof.

COROLLARY 2: let $\Sigma^n$ be a closed manifold, then the unit sphere bundle $(ST^*\Sigma, \xi)$ doesn't admit any subcritical fillings. (Weinstein filling)
Proof: look at $n$th homology &

$V$ is Weinstein of 2 dimensions less $\Rightarrow$ $H_n(V) = 0$

$H_n(ST^*\Sigma) \neq 0$ but

$H_n(W) = 0$ .

COROLLARY 3

Let $X^{2n}$ be a Liouville mfld then a nonempty composition of generalized Dehntwists along lagrangian spheres cannot be isotopic to identity - as a compactly supported symplectomorphism.

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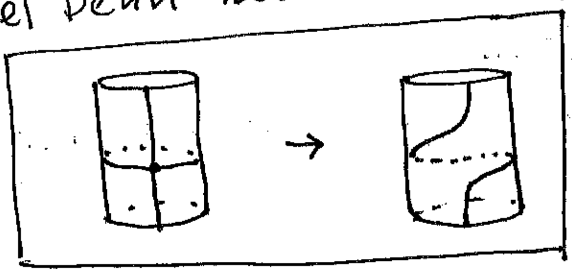
INTERLUDE ON HIGHER DIMENSIONAL DEHN TWISTS:

model Dehn twist on a surface $(n=1)$



is a compactly supported symplectomorphism of $T^*S^1$ which is nontrivial & acts as the antipodal map on $S^1$ = zero section (all powers are nontrivial) (NOT isotopic to identity)

(generalizes to higher dimension)

There exists a compactly supported symplectomorphism

$\tau : T^*S^n \to T^*S^n$ with the same properties.

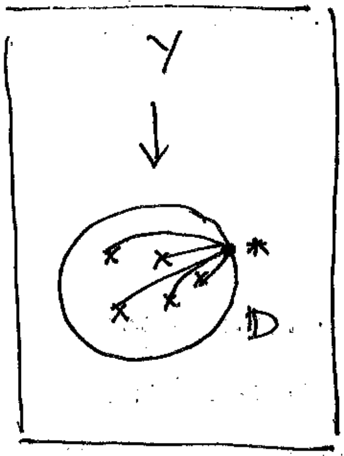
Hence, for any lagrangian embedding

$L : S^n \longrightarrow X^{2n}$ into a symplectomorphism $X^{2n}$

we get $\tau_L : X^{2n} \to X^{2n}$ on CSSymp by implanting

$\tau$ into a Weinstein nbd.

Given lagrangian embeddings $L_1, \dots, L_k$ into a Liouville domain $X^{2n}$, we can construct a Lefschetz fibration




fiber of $* = \mathcal{X}^{2n}$

vanishing cycles are $L_i$'s. (ordering matters)

Open book $(\mathcal{X}^{2n}, \tau_{L_1} \circ \cdots \tau_{L_k}) = \partial Y$ with smoothed corners.

This contact manifold has a weinstein filling int $Y$ with exactly $k$ critical handles.

$$\Rightarrow b_{n+1}(\underset{\overset{\shortparallel}{\text{int}(Y)}}{\text{filling}}) \geq k - \text{const}(\mathcal{X})$$

Proof: $(M, \xi) = OB(\mathcal{X}^{2n}, \text{Id})$
(of Corollary 3)

M is $\mathcal{X}^{2n}$-spliffable by definition.

$\Rightarrow$ $b_{n+1}$(any spherical filling) $< c$ for some constant

by BGZ thm.

$W = \tau_{L_1} \circ \tau_{L_2} \circ \cdots \circ \tau_{L_k}$ is isotopic to identity

$\Rightarrow$ any power of $W$ is also isotopic to identity

$\overset{\Rightarrow}{(\text{Elementary symp})} \rightsquigarrow OB(\mathcal{X}^{2n}, W^N) \xrightarrow[\text{contactomorphic}]{} OB(\mathcal{X}^{2n}, \text{Id})$

BUT this gives us a contradiction b/c

$OB(\mathcal{X}^{2n}, W^N)$ admits filling w/ arbitrarily large $b_{n+1}$, ~~but we know that~~

~~$b_{n+1}$(filling)~~ $b_{n+1} \geq Nk$ - constant.

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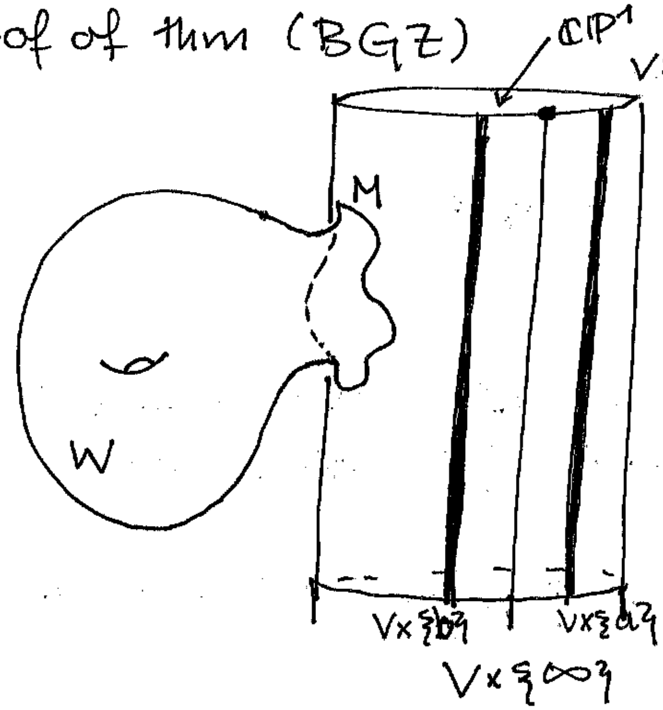
Proof of thm (BGZ)




$V\times\mathbb{CP}^1$ = cylinder.

$\tilde{Z} = Z \cup V\times\{\infty\}$ = everything

$Z$ = everything $-(V\times\{\infty\})$

$a, b \in \mathbb{CP}^1$

$M \hookrightarrow V\times\mathbb{D} \subset V\times\mathbb{C}$

we can replace interior by $W$ (i.e glue in)

choose a symplectic embedding of $\mathbb{D}\to\mathbb{CP}^1$ s.t.

$\mathrm{int}(\mathbb{D}) \to \mathbb{CP}^1 - \{\infty\}$

Choose an almost complex structure of the form

$J_{\tilde{Z}} = J_V \oplus i_{st}$ along $\tilde{Z} - \mathrm{int}(W)$

$J_V$ is admissable for $V \rightsquigarrow J_{\tilde{Z}}$ is also admissable for $V\times\mathbb{CP}^1$. (except $W$?)

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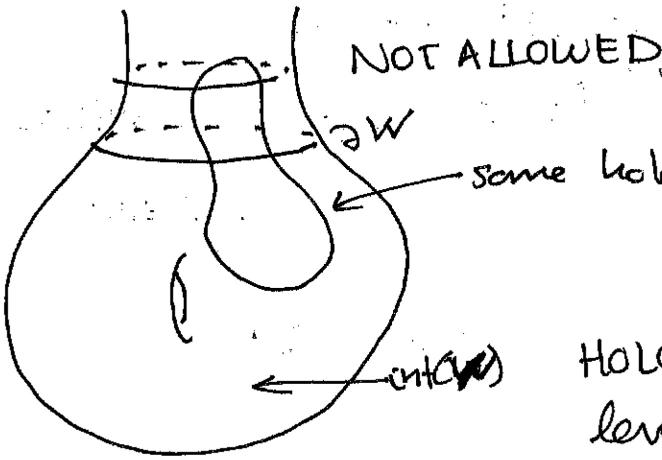
INTERLUDE ON MAXIMUM PRINCIPLES

If $W$ = finite type Liouville mfld, $\exists$ a class of acs (admissible) s.t.




HOLOMORPHIC CURVES cannot touch levels of the convex cone

$\Rightarrow$ $u: S^2 \to V$ is a hol curve & assume $u$ is not contained in $int(W)$

$\rightsquigarrow$ $u$ is constant

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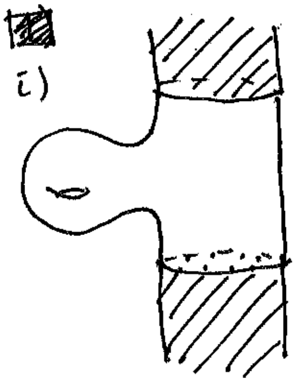
STRUCTURE LEMMA for holomorphic curves on $\tilde{Z}$

Assume V is of finite type. Let $u: \mathbb{C}P1 \to \tilde{Z}$

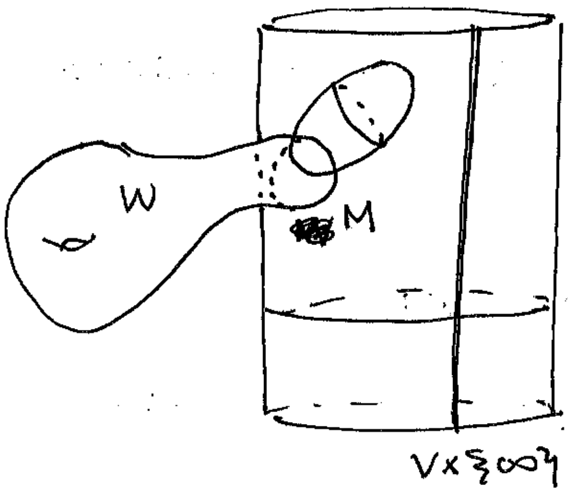
i)



shaded region = V looks like symplectization.

if u intersects the shaded region the u is a branched covering of $\{v\} \times \mathbb{C}P1$ for V in the shaded region.

ii)




Not allowed if u leaves W then it has to intersect $V \times \{\infty\}$ (max principle for $V \times \mathbb{C}$).

Let $\mathcal{M}$ be the moduli space

$$\mathcal{M} = \left\{ u : \mathbb{CP}^1 \to \tilde{Z} \;\middle|\; \begin{array}{l} \bullet\, [u] = [\{v\} \times \mathbb{CP}^1] \text{ for } v \text{ large enough} \\ \left.\begin{array}{l} \bullet\, u(-1) \in \{a\} \times V \\ \phantom{\bullet\,} u(1) \in \{b\} \times V \\ \phantom{\bullet\,} u(\infty) \in \{\infty\} \times V \end{array}\right\} \text{call this a slice condition.} \end{array} \right\}$$

Main PROPOSITION: • $\mathcal{M}$ is an oriented manifold of dimension $2n-2$.

• $\mathcal{M} \times \mathbb{CP}^1 \xrightarrow{ev} \tilde{Z}$ is proper and has degree 1.

<u>CLAIM</u>: $[\{v\} \times \mathbb{CP}^1]$ is simple.

Proof: Assume that $\underset{0^{\neq}}{[A]} + \underset{0^{\neq}}{[B]} = [\{v\} \times \mathbb{CP}^1]$.

where A & B are J-holomorphic.

positivity of intersection $[A] \cdot [\{\infty\} \times V] \geq 0$ &

$[B] \cdot [\{\infty\} \times V] \geq 0$

$\Rightarrow$ WLOG $[A] \cdot [\{\infty\} \times V] = 0$

This is impossible by positivity of intersections $\Rightarrow$ it doesn't intersect $\{\infty\} \times V$. $\Rightarrow$ [A] is contained in W. but W is a spherical filling.

PROPOSITION: $H_*(\mathcal{M} \times \mathbb{C}) \xrightarrow{ev} H_*(Z)$ is surjective

($ev_*\, ev : H_*(Z) \to H_*(Z)$ is an iso)

Now consider the diagram

$$H_* \left( \begin{tikzcd} \mathcal{M} \times \{1\} \arrow[r, "ev"] \arrow[d, "\cong"'] & V \times \{a\} \arrow[r] \arrow[d] & V \times \mathbb{C} - \mathrm{int}(W) \simeq M \arrow[dl] \\ \mathcal{M} \times \mathbb{C} \arrow[r, two heads, "ev"'] & Z \simeq W & \end{tikzcd} \right)$$

homotopy equiv

∎

More results from BGZ: $\dim(M) \geq 5$.

$(M, \xi)$ admits subcritical filling $W_0$ w/ homotopy type of a CW complex of dim $l_0 \leq n-1$

i) Let $W$ be a string aspherical filling of $M$

$$H_k(W) = \begin{cases} H_k(M) & k = 0, 1, \dots, l_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \quad H_k(W) = H_k(W_0)$$

ii) Assume $\pi_1(M) = 0$. Then for all string aspherical fillings of $M$ are diffeomorphic.