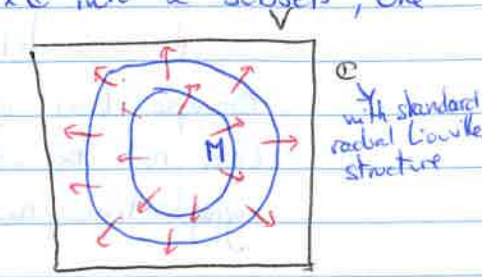
Umut - High dimensional J-holomorphic curve classification of fillings

Rem: the main point here is an application of the classification of fillings.

(M^{2n+1}, λ) connected contact manifold; $(V^{2n}, \omega, \mathcal{L})$ Liouville manifold ($\mathcal{L}$ = Liouville v.f.) that is of "finite type":

∂V modeled after symplectization.

We say M is V-spliffable if M is contactomorphic to a convex hypersurface $\tilde{M}$ in $V \times \mathbb{C}$ such that M divides $V \times \mathbb{C}$ into 2 subsets, one of which is modeled after the positive symplectization of M, i.e. in that component we have $\mathcal{L} \neq 0$, $\phi^t_{\mathcal{L}}(M)$ is the flow under $\mathcal{L}$, and $\bigcup_{t \geq 0} \phi^t_{\mathcal{L}}(M)$ cover it.



Theorem [Barth-Geiges-Zehmisch] (2016) Let W be an aspherical strong filling of M. Then, $\exists$ diagram

$$\begin{tikzcd} H_*(V) \arrow[rr, two heads] \arrow[dr] & \circlearrowleft & H_*(W) \\ & H_*(M) \arrow[ur] & \end{tikzcd}$$

and same for π_1.

M is V-spliffable.

Rem: $H_*(V) \twoheadrightarrow H_*(W)$ surjective implies that $H_*(M) \twoheadrightarrow H_*(W)$ is surjective as well.

Rem 1: for the surjection statement involving M and W only, we don't need to know what V is.

Theorem [Cieliebak, Cieliebak-Eliashberg] If M admits a subcritical Weinstein filling, then it is spliffable. (subcritical $\Rightarrow$ $\mathbb{C} \times$ smthg)

Rem 2: Barth-Geiges-Zehmisch actually prove a much stronger statement about the topological classification of fillings.

⚠ In high dimension, we don't get any information about symplectomorphism type.

Corollary [Eliashberg-Floer-McDuff] Let W be an aspherical filling of (S^{2n-1}, ξ_{std}), $n \geq 3$. Then, W is diffeomorphic to $\mathbb{D}^{2n}$.

Proof: S^{2n-1} is $\mathbb{C}^{n-1}$-spliffable. So $H_*(W) = 0$ in lots of dimensions, and with $\pi_1 = 0$, this implies (by smooth topology) the statement. $\square$

Note: BGZ proof is a careful generalization of EFM proof.

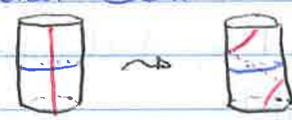
Corollary 2: Let Σ^n be a closed manifold. The unit sphere bundle $(ST^*\Sigma, \xi)$ doesn't admit any subcritical filling (Weinstein)

Proof: if it did, it would give an upper bound on other fillings, which is $H_n = 0$ (middle dimension: subcritical). But it does admit the unit disk bundle as a filling. $\boxtimes$

$\rightsquigarrow V$ is $(2n-2)$-Weinstein mfld $\rightsquigarrow$ no homology. $\square$

Corollary 3: Let X^{2n} be a Liouville manifold. Then, a nonempty composition of generalized Dehn twists along exact Lagrangian spheres can not be isotopic to the identity as a compactly supported symplectomorphism.

♪ Interlude ♪ on higher dimensional Dehn twists.

Model Dehn twist for $n=1$:  It is a compactly supported symplectomorphism of T^*S^1 which is non trivial, and acts as the antipodal map on $S^1 = 0$-section. (all powers are non trivial)

This generalizes to higher dimensions: $\exists\ \tau : T^*S^n \to T^*S^n$ compactly supported symplectomorphism with the same properties.

Hence, for any Lagr. embedding $S^n \xrightarrow{L} X^{2n}$ into a symplectic X^{2n}, we get $\tau_L : X^{2n} \to X^{2n}$ a compactly supported symplectomorphism by implanting τ into a Weinstein neighborhood.

Rem: construction of τ relies on the geodesic flow of S^n being periodic.

Given Lagr. embeddings $L_1, \dots, L_k$ into a Liouville domain X^{2n}, we can construct a Lefschetz fibration, with the fiber of $*$ being X^{2n}, and the vanishing cycles being the L_i's.

Y
$\downarrow$

$\mathbb{D} \simeq$  $* \rightsquigarrow OB(X^{2n}, \tau_{L_1} \circ \dots \circ \tau_{L_k}) = \partial Y$ (corners smoothed).

This has a Weinstein filling with k critical handles

$\Rightarrow b_{n+1}(\text{filling}) \geq k - \text{constant}(X)$

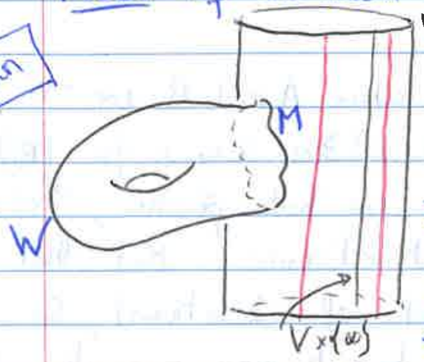
♪ End of interlude ♪

<u>Proof</u>: (of ~~main theorem~~ corollary 3) Let $(M,\xi) = OB(X^{2n}, id)$; then M is X^{2n}-splittable by definition. $\Rightarrow b_{n+1}(\text{aspherical filling}) < C$, by the main theorem.

Assume $w := \tau_{L_1} \circ \dots \circ \tau_{L_k}$ is isotopic to the identity $\Rightarrow w^N$ also. By elementary symplectic geometry, $OB(X^{2n}, w^N) \underset{\text{contacto}}{\simeq} OB(X^{2n}, id)$. But the LHS admits fillings with an arbitrarily large b_{n+1}, as $b_{n+1} \geq Nk - \text{constant}$ by the interlude. $\square$

<u>Proof of the main theorem</u>: $M \hookrightarrow V \times \mathbb{D}$ $(\subseteq V \times \mathbb{C})$; can replace the interior by W (ie glue in).

$\boxed{\dim \tilde{Z} = 2n}$



$\tilde{Z} = Z \cup V \times \{\infty\}$ the whole thing

Z = remove hypersurface at ∞

defined everywhere

Choose $\mathbb{D} \to \mathbb{CP}^1$ such that $int(\mathbb{D}) \to \mathbb{CP}^1 - \{\infty\}$.

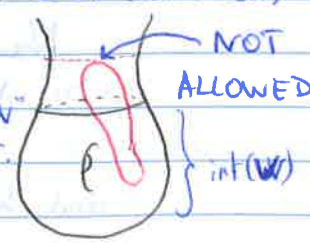
Choose an almost-$\mathbb{C}$-str. of the form $J_{\tilde{Z}} = J_V \oplus i_{st}$ along $V \times \mathbb{CP}^1 - int(W)$.

J_V is admissible for $V \rightsquigarrow J_{\tilde{Z}}$ is also admissible for $V \times \mathbb{CP}^1$.

Fix $a, b \in \mathbb{CP}^1$, and add (in red) the hypersurfaces $V \times \{a\}$ and $V \times \{b\}$.

♪ <u>Interlude</u> ♪ on maximum principles.

If W is a finite type Liouville manifold $\exists$ class of a-$\mathbb{C}$-str (admissible) such that holomorphic curves can not touch levels of the convex cone. In other words; if $u: S^2 \to W$ is a holomorphic curve that is not contained in $int(W)$, then it's <u>constant</u>.



♪ <u>End of interlude</u> ♪

<u>Rem.</u> we have a max principle for V, and for $V \times \mathbb{CP}^1$. We'll use both.

//// = V looks like symplectization there.

1)

If u intersects the shaded region, then u is a branched covering of $\{v\} \times \mathbb{CP}^1$ for v is the shaded region, by maximum principle for V.

2)

hyperplane at ∞.

If u leaves W, then it has to intersect $V \times \{\infty\}$, by the maximum principle for $V \times \mathbb{C}$.

Let $\mathcal{M}$ be the moduli space $u : \mathbb{CP}^1 \to \tilde{Z}$ such that

i) $[u] = [\{v\} \times \mathbb{CP}^1]$ for v large enough.

ii) $u(-1) \in \{a\} \times V$, $u(1) \in \{b\} \times V$, $u(\infty) \in \{\infty\} \times V$.

Rem: Every curve in $[\{v\} \times \mathbb{CP}^1]$ has to intersect all 3 $\{a\} \times V$, $\{b\} \times V$ and $\{\infty\} \times V$; the condition ii) is just a slicing condition so we don't have to mod out by anything.

Main proposition: * $\mathcal{M}$ is an oriented manifold of dim $2n-2$.

* $\mathcal{M} \times \mathbb{CP}^1 \xrightarrow{ev} \tilde{Z}$ is proper and degree 1.

Claim: $[\{v\} \times \mathbb{CP}^1]$ is simple.

Proof: assume that $[\{v\} \times \mathbb{CP}^1] = [A] + [B]$ where A and B are J-hol.
By positivity of intersections, $[A] \cdot [\{\infty\} \times V] \geq 0$ (same for $[B]$)
$\Rightarrow$ WLOG $[A] \cdot [\{\infty\} \times V] = 0$: if they were both positive, their sum would intersect twice, but it intersect once. But that intersection # is the geometric one (positivity of intersections), so it does not intersect ∞, so it has to be constant by max. principle. $\square$

Proposition: $H_*(\mathcal{M} \times \mathbb{C}) \xrightarrow{ev} H_*(Z)$ is surjective.

Proof: $ev_*(ev^!) : H_*(Z) \to H_*(Z)$ is an iso. $\square$
↳ integration along fibers

take H_* everywhere

Now, consider the diagram

$$\begin{tikzcd} \mathcal{M} \times \{1\} \arrow[r, "ev"] \arrow[d, hook, "\cong"'] & V \times \{a\} \arrow[r] \arrow[d, hook] & V \times \mathbb{C} - \operatorname{int} W \arrow[dl] \\ \mathcal{M} \times \mathbb{C} \arrow[r, two heads, "ev", "\text{Surj by Prop}"'] & Z & \end{tikzcd}$$

$Z \cong W$, $V \times \mathbb{C} - \operatorname{int} W \cong M$

So $V \times \{a\} \to Z$ has to be surjective,

and so $\underset{\cong M}{V \times \mathbb{C} - \operatorname{int} W} \to \underset{\cong W}{Z}$ also.

So, $H_*(M) \to H_*(W)$ is surjective. $\square$

Rem: $ev : \mathcal{M} \times \{1\} \to V \times \{a\}$ lands in $V \times \{a\}$ by the slicing condition ii) above.