Agustín - Strongly fillable contact manifolds & J-holomorphic foliations (Wendel paper)

- A contact manifold (M, ξ) is planar if it admits an OBD supporting ξ with genus zero pages.
- A Lefschetz fibration is allowable if all vanishing cycles are homologically nontrivial
- If $\pi : M \setminus B \to S^1$ OBD, denote $\hat{\pi} : M \setminus N(B) \to S^1$,

  $N(B) = \text{nhd}(B)$

DEFINITION: Given an OBD, $\pi : M \setminus B \to S^1$, α, a contact form, is Giroux if $d\alpha|_{page} > 0$ and if α is positive on the binding (so binding is closed Reeb orbit & Reeb v.f transverse to the pages)

We say $\xi = \ker \alpha$ is supported by the OBD

$\boxed{\text{THM 1}}$ Suppose (W, ω) is a strong symplectic filling of a planar contact manifold (M, ξ) and $\pi : M \setminus B \to S^1$ planar OBD supporting ξ, then $\exists$ some enlarged version of (W, ω) obtained by attaching trivial symplectic cobordism to W, such that W admits a symplectic lefschetz fibration

$$\pi : W' \to \mathbb{D} \quad \text{s.t.} \quad \pi'|_{\partial W'} = \hat{\pi}$$

Moreover $\pi : W' \to \mathbb{D}$ is allowable if W is minimal.

$\boxed{\text{Corollary 1}}$ Every strongly fillable planar contact mfld is stein fillable.

from Eliashberg's characterization of stein mflds.

minimal filling = No exceptional curves.

Strongly but not stein fillable $\Rightarrow$ Not planar.

THEOREM 2: Suppose I have a strong symplectic filling of $(\mathbb{T}^3, \xi_0)$

$$\xi_0 = \ker(\cos(2\pi\theta)\, dq_1 + \sin(2\pi\theta)\, dq_2)$$

Then one can attach a trivial symplectic cobordism such that the enlarged filling admits a symplectic LF

$$\Pi : W' \longrightarrow [0,q] \times S^1$$

for which $\Pi|_{\partial W' \setminus N(Z)} = \hat{\Pi}_0$

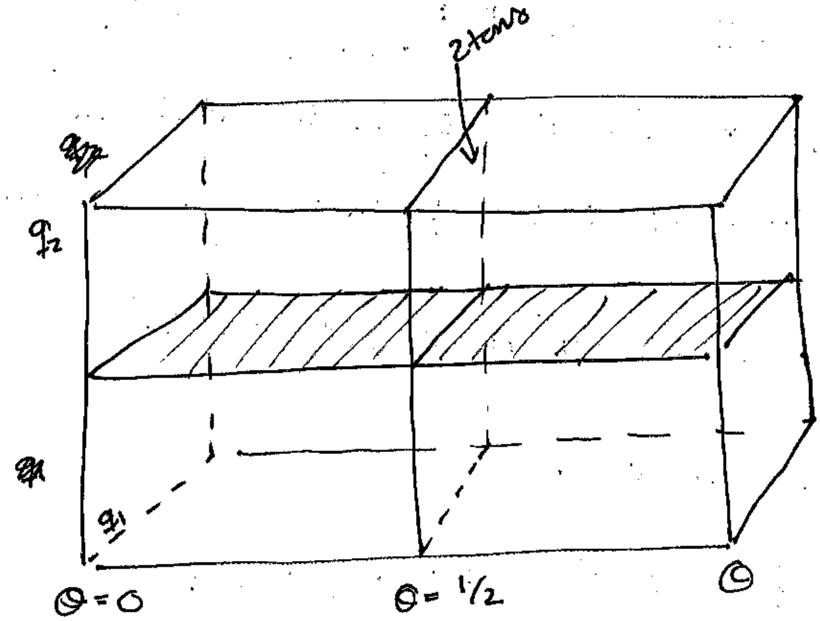
Moreover, every singular fiber of LF is the union of an annulus w/ an exceptional sphere.
In particular, if (W, ω) is minimal there are no singular fibers.

$$\mathbb{T}^3 = S^1 \times S^1 \times S^1 = \underset{(q_1, q_2, \theta)}{\mathbb{T}^2 \times S^1}$$

$$Z = \{\theta \in \{0, 1/2\}\} \subseteq \mathbb{T}^3$$

$$\Pi_0 : \mathbb{T}^3 \setminus Z \longrightarrow \{0,1\} \times S^1$$

$$(q_1, q_2, \theta) \longmapsto \begin{cases} (0, q_2) & \text{if } \theta \in (0, 1/2) \\ (1, q_2) & \theta \in (1/2, 1) \end{cases}$$



THM 4: All minimal strong fillings of $\mathbb{T}^3$ are symplectically deformation equivalent & every exact filling of $\mathbb{T}^3$ is symplecto to a star shaped domain in $(T^*\mathbb{T}^2, \omega_0)$

i.e $\{(q, tf(q,p)p) \in T^*\mathbb{T}^2 \mid t \in [0,1], (q,p) \in S^*\mathbb{T}^2$
$f: S^*\mathbb{T}^2 \to (0,\infty)\}$

COROLLARY Every minimal strong filling of $\mathbb{T}^3$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{D}^2$

THM 5: The grp $\mathrm{sym}_c(T^*\mathbb{T}^2, \omega_0)$ of compactly supported symplecto's is contractible.

$\boxed{\text{DEFINITION:}}$ A finite energy foliation $\mathcal{F}$ on some mfld (M_0, λ, J) where λ = contact form
J = λ compatible ACS

is a foliation of $\mathbb{R} \times M_0$ s.t.

- $\forall u \in \mathcal{F}$ leaf, any $\mathbb{R}$-translation of u is also a leaf.
- Every $u \in \mathcal{F}$ is the image of a finite energy J-hol curve satisfying some uniform energy bound.

$\boxed{\text{DEFN}}$ A leaf $\in \mathcal{F}$ is interior if it is not a trivial cylinder & all its ends belong to Morse Bott submflds in M_0

$\boxed{\text{DEFN}}$ $\mathcal{F}$ is positive if every leaf which is not a trivial cylinder has only positive ends.

$\boxed{\text{DEFN}}$ $u \in \mathcal{F}$ is stable if it has genus zero all punctures are odd and Fred index$(u) = 2$.

$\boxed{\text{DEFN}}$ $u \in \mathcal{F}$ is asymptotically simple if all of its asymptotics are simply covered & belong to pairwise disjoint Morse Bott families.

Holomorphic curves & compactness

(M^3, ξ) with Morse-Bott λ, J_+ λ-compatible

$M_0^\xi \subseteq M$ compact submfld w/ Morse Bott boundary

$\mathcal{F}_+$ positive finite energy foliation on $(M_0, \lambda J_+)$ containing an interior stable leaf that is asymptotically simple

(W^∞, ω) ~~compact~~ non compact mfld s.t.

$$W^\infty = W \cup_{\partial W} ([R, \infty) \times M) \text{ for } R \in \mathbb{R}$$

$$\omega|_{[R,\infty)\times M} = d(e^a \lambda)$$

$$\overline{W}^\infty = W \cup_{\partial W} ([R, \infty) \times M)$$

$$\partial \overline{W}^\infty = M$$

Choose J on W^∞ which is ω-compatible such that

$$J|_{[a_0,\infty)\times M} = J_+ \quad \text{for } a_0 \in [R, \infty)$$

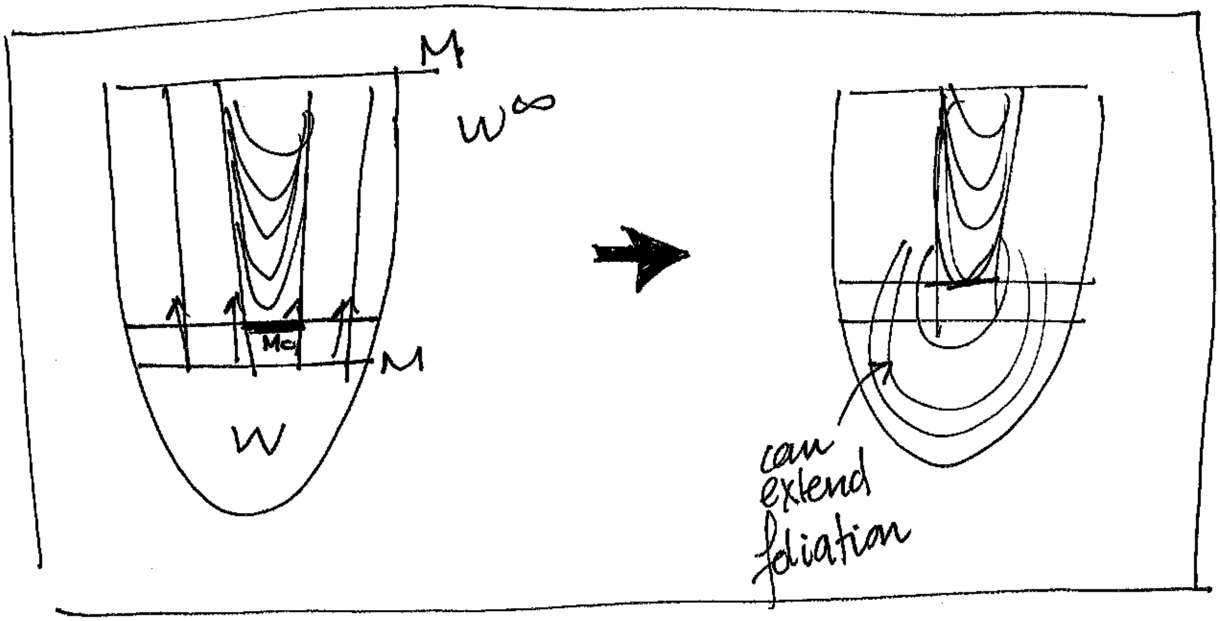
$$\mathcal{F}_0 = \bigcup \text{ leaves in } \mathcal{F}_+ \text{ lying in } [a_0, \infty) \times M$$

~~You can't assume J_+ is generic. You~~

$\mathcal{M}$ = moduli space of finite energy J-curves in W

$\overline{\mathcal{M}}$ = compactification of $\mathcal{M}$.

Fix $u_0 \in \mathcal{F}_0$ as simple curve, let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the connected component containing u_0.


M
W^∞
M_a
M
W
can extend foliation