Roberta - Pseudo-holomorphic sh*t

Why? Proposition: given (M,ω), $\exists$ J compatible almost-$\mathbb{C}$-structure. Moreover, the space of those is contractible.

Uses of J-hol. curves:

- Pick a J and use it to construct some curves, and directly use these curves (ex: proof of OT $\Rightarrow$ non-fillable)
- Use J to build invariants of (M,J); check independance on J, so we get invariants of (M,ω).

Definition: given (M,J) almost complex, a J-hol curve is a map $u:(\Sigma,j)\to(M,J)$, where (Σ,j) is a Riemann surface, such that $J\circ du = du\circ j$

Good properties:

* Unique continuation: if $u_1(z)=u_2(z)$ and all derivatives agree, then $u_1=u_2$
* $\mathrm{Crit}(u)$ is finite
* Either (a) u factors through a branched holomorphic n-cover of Σ_g, or (b) the set of points where u is injective is open and dense in Σ_g.

When it is (b), we say that u is simple. So, a J-hol curve always factors through a branched cover followed by a simple curve.

Moduli space of J-hol. curves:

Can we deform u to get something J-hol? In how many directions? Convergence of sequences?

Definition $\mathcal{M}^*(M,J) := \{$all simple J-hol. curves$\}$. We can add restriction. $\mathcal{M}(A,J)=\{u:\Sigma\to M,\ J\text{-hol} \mid u_*[\Sigma]=A\}$, for $A\in H_2(M)$.

Transversality: is $\mathcal{M}(A,J)$ a manifold? We can see it as the 0-level set of the functional $F(u) = J\circ du - du\circ j$, defined on $C^\infty(\Sigma,M)$. $\hookrightarrow$ is 0 a regular value of F?

Intuition from finite-dim geometry: up to deforming f, 0 can be made a regular value. In our case, for generic J, 0 is a regular value, which means $\mathcal{M}^*(A,J)$ is a manifold.

⚠ $\mathcal{M}^*(A,J)$ = simple curves in $\mathcal{M}(A,J)$.

(Virtual) dimension of $\mathcal{M}^*(A,J)$: when J is regular,

$\dim \mathcal{M}^*(A,J) = \dim(\ker(D_uF))$ (Fredholm index)

$= \mathrm{ind}(D_uF)$, since we assume D_uF is surjective

$= n(2-2g) + 2c_1(A)$ $(\dim M = 2n)$

If not transverse, we still call this number the virtual dimension.

Automatic transversality: no need to modify J.

* dim = 4
* not all of $\mathcal{M}^*(A,J)$; only in a neighbourhood of $u \in \mathcal{M}^*(A,J)$, or some subset $S \subseteq \mathcal{M}^*(A,J)$.

Theorem [McDuff] if J is integrable in a nbhd of the image of u in M and $c_1 > 2(g-1)$, then u is a regular point. ~~Here $c_1 = c_1(\mathcal{V}_H(\mathrm{im}\,u))$~~. And u must be ~~simple~~ embedding.

Theorem [Wendl]: $u: (\Sigma_g, \text{some punctures}) \to M$ has ends on non-degenerate Reeb orbits (of the boundary), then u is regular if $2g - 2 + h_+ <$ Fredholm index.

h_+ → # positive punctures

Reparametrization:

$\mathcal{M}(A,J) := \{\text{all couples } (\Sigma_g, u: \Sigma_g \to M)\} / G$, $G = \{\text{autom of } \Sigma_g\}$

So, $u_1 \sim u_2$ if $u_1 = u_2 \circ f$ for $f: \Sigma_g \to \Sigma_g$ biholomorphic.

Evaluation map: we can't define it on $\tilde{\mathcal{M}}$ since we reparametrize. Instead, define it on $(\mathcal{M}(A,J) \times \Sigma_g)/G$, as

$ev(u,z) := u(z)$, and the action of G is $(u,z) \sim (u \circ f^{-1}, f(z))$

So now, can study subspaces

$\mathcal{M}(A,J,z_1,\dots,z_k) = \{\text{non-param. curves going through } z_1,\dots,z_k \in M\}$

ex: Gromov-Witten invariants:

$GW(A,J,z_1,\dots,z_k) = \#\mathcal{M}(A,J,z_1,\dots,z_k)$, when $\dim \mathcal{M}(\dots) = 0$.

Rem: only works if $\tilde{\mathcal{M}}(\dots)$ is also compact.

Compactness [Gromov]

What do sequences in $\tilde{\mathcal{M}}^{M}(A, J)$ approach?

Sequences can diverge, if the "energy" goes to ∞, where

$E(u) = \int_\Sigma u^*\omega$

<u>Rem</u>: in $\tilde{\mathcal{M}}(A,J)$, the energy is fixed, as ω is closed.

If the energy is bounded, we can say that there is a "convergent" subsequence, where the limiting curve is allowed to have bubbles:

$u: \Sigma_g \cup \mathbb{CP}^1 \cup \ldots \cup \mathbb{CP}^1 \to M$

So $\tilde{\mathcal{M}}(A,J)$ is not compact, but:

Theorem: $\tilde{\mathcal{M}}(A,J) \cup \{\text{all } u: \Sigma_g \cup \mathbb{CP}^1 \cup \ldots \cup \mathbb{CP}^1 \to M\}$ is compact.

<u>Rem</u>: there are versions of this with punctures, with point constraints, ...

Positivity of intersections:

Let u_1, u_2 be J-hol. in M^4.

Theorem [Gromov - McDuff] The multiplicity of intersection is a positive number at each intersection point.

Corollary: $C_1 \cdot C_2 = 0 \Leftrightarrow$ they are disjoint.