Emily - The structure of ruled & rational symplectic 4-mflds

(paper by McDuff).

①

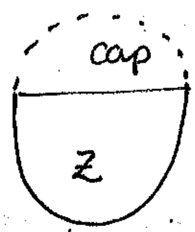
GOAL: Classify compact symplectic 4-manifolds containing a symplectically embedded S^2, C, with nonnegative self intersection

Why we care?

If you want to classify fillings of Z of X

add a cap ~~cap~~

& now classify compact mflds w/ such a cap.



X contact

Containing Cap $\Rightarrow$ strong restrictions on $V \Rightarrow$ strong restrictions on Z.

TERMINOLOGY:

(V, ω) compact smooth symplectic 4-manifold

C rational curve = symplectically embedded S^2

an exceptional curve is a rational curve C s.t. $C \cdot C = -1$.

(V, C, ω) minimal if $V \setminus C$ contains no exceptional curves.

V ruled = fibered by rational curves.

THM (McDuff '90) (V, C, ω) minimal and $C \cdot C \geq 0$, then

(V, ω) symplectomorphic to

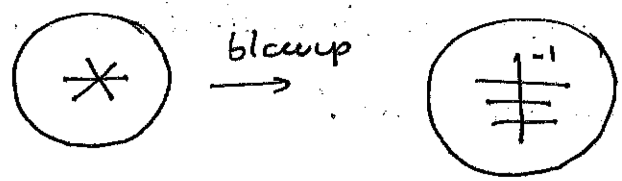
(1) $(\mathbb{C}P^2, \omega_{FS})$

(2) symplectic S^2-bundle over compact M. Moreover C

is taken to
 (i) complex line or quadric
 (ii) fibre or section of bundle

REMARK: C has positive self intersection.
we are only classifying minimal pairs.

THM: Every (V, C, ω) covers a minimal $(\overline{V}, \overline{C}, \overline{\omega})$ obtained by blowing down a finite collection of exceptionals in $V \setminus C$. Given these $\overline{\omega}$ unique up to isotopy.



is also
$M \# \overline{\mathbb{C}P^2}$

<u>Upshot</u>: A classification problem boils down to minimal case.

MAIN TOOL: adjunction formula

S is a rational holomorphic embedded curve.

$$TV|_S = TS \oplus \nu S$$

$$c_1(TV|_S) = c_1(TS) + c_1(\nu S)$$

$$\boxed{c_1(V)\cdot[S] = \chi(S) + [S]\cdot[S]}$$ ADJUNCTION FORMULA

Mega lemma: $\exists$ tame J s.t. $[C]$ may be represented by a J-holomorphic cusp curve

$$S = S_1 \cup \dots \cup S_m \quad \text{where } [S_i] = A_i$$

is J-indecomposable and J is regular for all A_i-curves
Moreover, the S_i are distinct embedded curves with self intersection $-1, 0$ or 1 and there is at least one component with $A_i \cdot A_i \geq 0$.

J-indecomposable: A_i can't be split up more.

Proposition 1: If F is a rational B-curve in (V,ω) where B is simple & $B\cdot B=0$, then $\exists\ \pi: V \to M$ compatible with ω, with F a fibre (ω nondegenerate on fibres).

Proof: Take a tame almost complex structure, J, that splits near F. If f is a J-holomorphic parametrization of F then $c_1(\nu F)=0$

(f,J)

Automatic transversality tells us that ~~■~~ is regular

B simple $\Rightarrow \mathcal{M}_{0,1}(J,B)$ compact

genus one marked point

$$f: S^2 \longrightarrow V$$
$$p^{\varepsilon}$$

$K = \#$ of marke points

Then $\dim = 2n + 2c_1(C) + 2K - \underset{\dim(PSL(2,\mathbb{C}))}{6}$.

$$= 4 + 2\Lambda(C) + 2[C]\cdot[C] + 2k - 6$$
$$= 4 + 2(2+0) + 2\cdot 1 - 6 = 4$$

Consider the evaluation map

$$ev: \mathcal{M}_{0,1}(J,B) \longrightarrow V$$

$B\cdot B = 0 \Rightarrow$ there is at most one B-curve through each point in V
$\Rightarrow$ degree of ev is ≤ 1

family of such J-holomorphic curves in a nhd of F, there is a

Then degree at least 1. (by regularity)

$\Rightarrow$ The degree is exactly one

$\Rightarrow$ There is exactly one J-holomorphic B-curve through every point of V.

These form fibres of continuous surjection

$$\pi : V \to M$$

Fibres are holomorphic $\Rightarrow$ symplectic so compatible.

<u>REMARK</u>: ω compatible with π $\Rightarrow$ ω determined up to isotopy by its cohomology class $[\omega] \in H^2(V)$.

PROPOSITION 2: (V, C, ω) are minimal and $C \cdot C \geq 0$

(i) If $S_i \cdot S_i = 1$ for some $i \Rightarrow V = \mathbb{CP}^2$ and $m = 1$ or 2

(ii) If $S_i \cdot S_i = -1$ for some $i \Rightarrow V \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and there is one such i and all other S_j are homologous with $S_j \cdot S_j = 0$

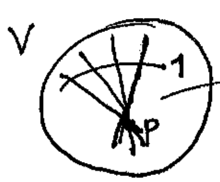
(iii) If $S_i \cdot S_i = 0$ for all $i \Rightarrow$ all S_i except maybe one are homologous and V is an S^2-bundle.

Comments on proof:

(i) Gromov showed this for $m = 1$.

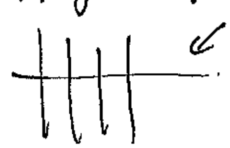
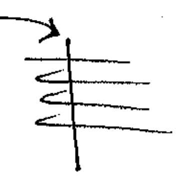
Consider $ev : \mathcal{M}_{0,2}(J, A) \to V \times V$

show $\deg = 1 \Rightarrow \exists !$ J-holomorphic curve A thru each pair of points



map to $\mathbb{CP}^2$

(ii) apply adjunction to get a contradiction

(iii)  the one that is not homologous 

By Propositions 1 & 2, we get the diffeomorphism types & symplectomorphism type determined just by $[\omega]$ up to isotopy.

COROLLARY: Diffeomorphism type is determined by $p = C \cdot C \geq 0$ for $p \neq 0$ or 4.

$\boxed{p=0} \overset{\text{prop 1}}{\Longrightarrow}$ S^2-bundle

$\boxed{p=1} \overset{\text{Gromov}}{\Longrightarrow}$ $\mathbb{CP}^2$, C is sent to a line.

$\boxed{p \geq 2} \Rightarrow$

$$p = C \cdot C = (A_1 + \dots + A_m)^2 = \sum_i A_i^2 + \sum_{i<j} 2 A_i \cdot A_j$$

In case (ii) $\quad p = -1 + \sum_{i<j} 2 A_i \cdot A_j \in 2\mathbb{Z}+1$

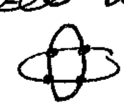
case (iii) $\quad p = 0 + \sum_{i<j} 2 A_i \cdot A_j \in 2\mathbb{Z}$

If p is odd $\Rightarrow V \simeq \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (diffeo)

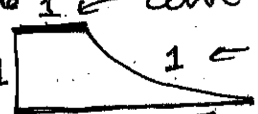
If p is even & $p \geq 2$ $\Rightarrow V \simeq S^2 \times S^2$

If $p = 4 \Rightarrow V \simeq S^2 \times S^2$ with $C \mapsto \Gamma_2$

OR $V \simeq \mathbb{CP}^2$, with $C \mapsto Q$



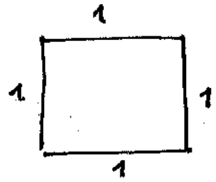
Hirzebruch surface



$\mathbb{F}_p \simeq \mathbb{F}_{p+2} \quad \forall p.$

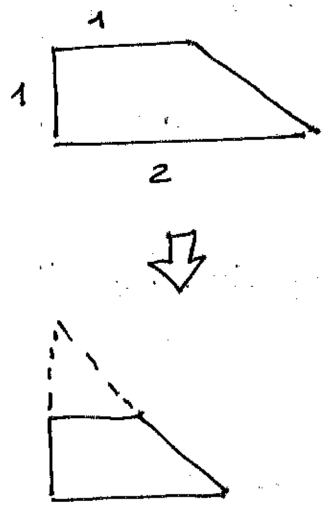
no two are symplectomorphic

$\mathbb{F}_0$



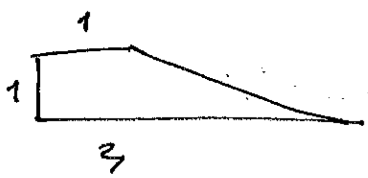
$S^2 \times S^2$

$\mathbb{F}_1$



$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

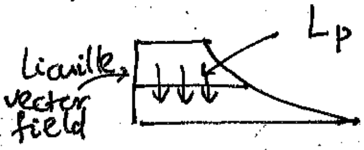
$\mathbb{F}_2$



Implication on fillings:

Get $\mathbb{F}_p$ by gluing $\mathcal{O}(p)$ to $\mathcal{O}(-p)$

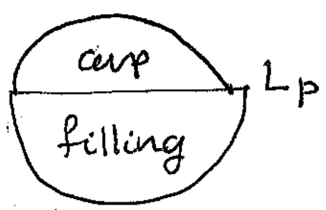
CLAIM: $\partial(\mathcal{O}(p)) = L_p = L(p,1)$



L_p as a contact type hypersurface

$\partial(\mathcal{O}(-p)) = L_p$ (opposite orientation)

<u>Punchline</u>:



cap $= \mathcal{O}(p)$

$\Rightarrow$ it contains curve w/ self intersection p

$\Rightarrow$ McDuff shows that minimal such compact mflds are determined uniquely up to symplectomorphism if we fix $[\omega]$ and up to diffeo for $p \neq 0, 4$

$\Rightarrow$ L_p have minimal symplectic fillings for $p \neq 4$ these are unique up to diffeo. If fix $[\omega]$ unique up to symplecto (L4 has exactly 2 nondiffeo fillings).