Roberta: PSEUDO-HOLOMORPHIC SH*T. 21/May/2017

Why do we care?

PROPOSITION: Given a symplectic manifold (M, ω) there exists a compatible almost complex structure J & moreover the space of compatible J's is contractible.

Morally: we can study ω via J.

Uses of J-holomorphic curves:

Pick a J & use it to construct curves,

& directly use these curves.

Use J to build invariants of (M, J). check independence on J $\Rightarrow$ gives us invariants of (M, ω).

DEFINITION: Given (M^{2n}, J) an almost complex manifold, a J-holomorphic curve is a map

$$u : (\Sigma, j) \longrightarrow (M^{2n}, J)$$

where $(\Sigma_g, j) = $ (Riemann surface of genus g, j) s.t.

$$J \circ du = du \circ j \qquad \text{(like a holomorphic map)}$$

OJO: J-holomorphic curve = pseudo-holomorphic curve

GOOD PROPERTIES OF J-HOLOMORPHIC CURVES:

- Unique continuation
  ($u_1(z) = u_2(z)$ & all derivatives agree $\Rightarrow$ $u_1 = u_2$)

- $\operatorname{crit}(u)$ is finite

- Either
  1) u factors through a (branched) holomorphic n-cover of Σ_g OR
  2) set of points where u is injective is open & dense in Σ_g.

if (2) then we call it a simple curve.

The (Moduli) space of J-holomorphic curves:

we care about how these curves fit together? or
can we deform u and still get a pseudo holomorphic curve?
in how many directions can we deform u?
can we look at a sequence of such curves & study convergence of such a sequence?

DEFINITION: $\mathcal{M}^*(M, J) = \left\{ \begin{array}{c} \text{all simple} \\ J\text{-holomorphic curves} \end{array} \right\}$

Q: Is this space connected, compact, open, with boundary?

Add a restriction:

DEFINITION: $\mathcal{M}(A, J) = \left\{ \begin{array}{c} u : \Sigma_g \to M,\ J\text{-holomorphic s.t.} \\ u_*[\Sigma] = A \end{array} \right\}$

for $A \in H_2(M)$

$\boxed{\text{TRANSVERSALITY}}$ Is $\mathcal{M}(A, J)$ a manifold?

Idea: $\mathcal{M}^*(A, J) = \{ u : \Sigma \to M \mid \underbrace{J \cdot du - du \cdot j = 0}_{\text{functional } F(u) = 0} \}$

so $\mathcal{M}(A, J)$ is a level set of $\left\{ \begin{array}{c} \text{smooth maps} \\ f : \Sigma \to M \end{array} \right\}$

Now question is:

Is 0 a regular value of F?

Intuition (from finite dimensional geometry) up to deforming f, 0 can be made a regular value.

THEOREM For a generic J, 0 is a regular value

$\Rightarrow \mathcal{M}^*(A, J)$ is a manifold.

(Virtual) dimension of $\mathcal{M}^*(A, J)$ when J is regular, then

$$\dim \mathcal{M}^*(A, J) = \dim(\operatorname{Ker}(D_u F)) = \underbrace{\operatorname{Fred}(D_u)}_{\text{Fredholm index}}$$

(assume D_u is onto)

$$\mathrm{Fred}(D_u) = n(2-2g) + 2c_1(A)$$

This # is defined even if we do not have transversality but is then not nec = dimension.

AUTOMATIC TRANSVERSALITY: No need to deform J

- dim = 4
- Not all of $\mathcal{M}^*(A,J)$, only in a nhd of a point $u \in \mathcal{M}^*(A,J)$ or some subset $S \subseteq \mathcal{M}^*(A,J)$.

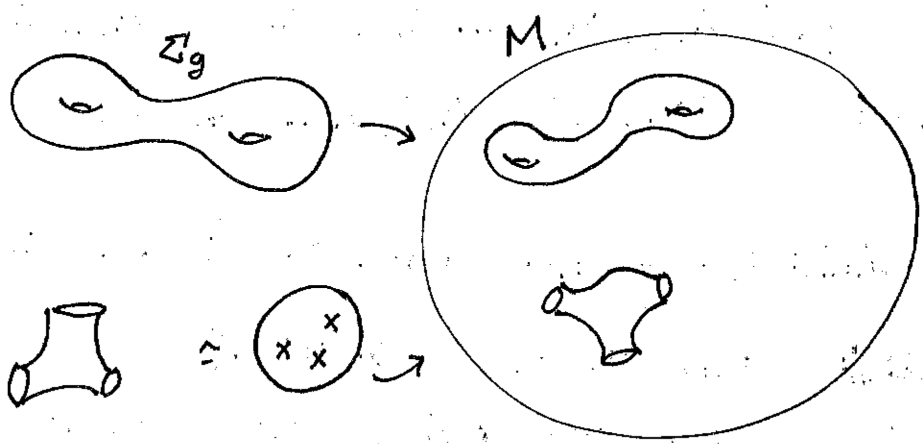
THEOREM (McDuff) Assume u is an embedding. If J is integrable in a neighborhood of $\mathrm{Im}(u) = \underset{\text{curve}}{C} \subset M$ and $c_\nu > 2(g-1) \Rightarrow u$ is a regular point

- $c_\nu = c_1(TM|_C) = c_1(M) - [C]$.

<u>Wendl</u>: M = symplectization

$u: (\Sigma_g$, some punctures$) \to M$ has ends on nondegenerate Reeb orbits $\Rightarrow u$ regular if $2g - 2 + n_+ <$ Fred index

n_+ = # of positive punctures.



there is a problem of reparametrization;

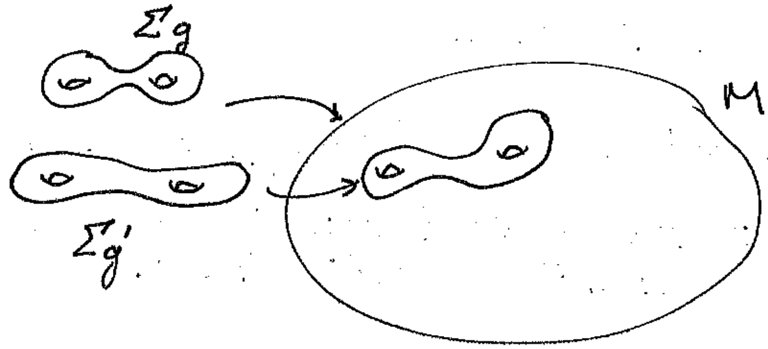
Consider $\widetilde{\mathcal{M}}(A,J) = \underbrace{\{(\Sigma_g, u: \Sigma_g \to M)\}/G}_{= \mathcal{M}(A,J)/G}$ all caples

$G = \left\{ \begin{array}{l} \text{automorphisms} \\ \text{of } \Sigma_g \end{array} \right\}$

so $\quad U_1 \sim U_2$ if

$U_1 = U_2 \circ f$

$f : \Sigma_g \to \Sigma_g$ holomorphic.



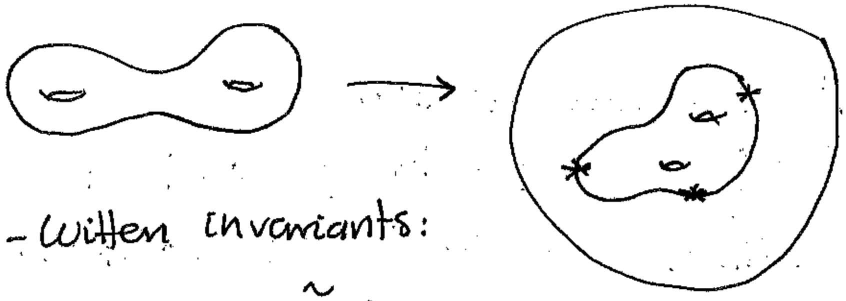
Evaluation Map:

Can't define on $\widetilde{\mathcal{M}}$, so instead we can define it on

$(\mathcal{M}(A, J) \times \Sigma_g) / G$

$ev(U, z) = U(z)$ well defined b/c $\quad (U, z) \sim (U \circ f^{-1}, f(z))$.

$\mathcal{M}(A, J, z_1, \ldots, z_k) = \left\{ \begin{array}{c} \text{nonparametrized curves going thru} \\ z_1, \ldots, z_k \in M \end{array} \right\}$



EXAMPLE: Gromov-Witten invariants:

$$GW(A, J, z_1, \ldots, z_k) = \# \widetilde{\mathcal{M}}(A, J, z_1, \ldots, z_k)$$

provided $\dim(\widetilde{\mathcal{M}}(A, J, z_1, \ldots, z_k)) = 0$

REMARK: Really only works if $\widetilde{\mathcal{M}}(\ldots)$ is also compact.

COMPACTNESS (GROMOV)

Is $\widetilde{M}$ compact ie what do sequences in $\widetilde{\mathcal{M}}(M, J)$ approach?

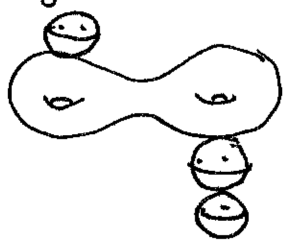
- Sequence can diverge if "energy" goes to ∞ and

$$E = \int_{\Sigma_g} u^*(\omega)$$ where ω is symp form compatible w/ J

(in $\tilde{\mathcal{M}}(A,J)$ energy is fixed).

- Otherwise (if energy is bounded) there is a convergent subsequence, the limiting curve is allowed to have bubbles

$u : \Sigma_g \cup \mathbb{CP}^1 \cup \dots \cup \mathbb{CP}^1 \to M$


$\to M$

So $\tilde{\mathcal{M}}(A,J)$ is not compact

THEOREM $\tilde{\mathcal{M}}(A,J) \cup \left\{ \begin{array}{l} \text{all} \\ u : \Sigma_g \cup \mathbb{CP}^1 \cup \dots \cup \mathbb{CP}^1 \to M \end{array} \right\}$ is compact.

Given u_1, u_2 J holomorphic in M of dimension 4

Gromov-McDuff: the multiplicity of intersection is a positive number at each intersection point.

COROLLARY: $C \cdot C = 0 \Leftrightarrow$ they are disjoint