(24)

Emily - McDuff's rational ruled classification

Plan: 1) Motivation and context
2) Ideas of proof
3) An application to fillings

1) Aim: classify compact symplectic 4-manifolds containing a symplectically embedded copy of S^2, C, with nonnegative self-intersection.

Why? Fillings! To classify fillings Z of X

$\updownarrow$

Classify compact manifolds V containing a cap

Containing $C \Rightarrow$ strong restrictions on what V can be

$\Rightarrow$ " " " " Z can be

(the cap contains the 2-sphere C)

Terminology: (V, ω) compact smooth symplectic 4-manifold.

C rational curve (i.e. an symplectically embedded S^2)

An exceptional curve is a rational curve with self-intersection -1.

We call (V, C, ω) minimal if $V \setminus C$ contains no exceptional curve.

(V, ω) is rational ~~ruled~~ if it contains a rational curve, and ruled if it's fibrated by rational curves.

Theorem: [McDuff '90] If (V, C, ω) is minimal, and $C \cdot C \geq 0$, then (V, ω) is symplectomorphic to (1) $(\mathbb{CP}^2, \omega_{FS})$

(2) a symplectic S^2-bundle over compact M.

Moreover, C is taken to (1) complex line or quadric

(2) fibre or section of bundle.

Rem: * easy to have negative curves, by blowing up

* only classify minimal pairs. Actually, if $V \setminus C$ contains a (-1)-curve, we can blow it down:

Recall blow up: $(*)\ M \longrightarrow (\ne)\ M \# \overline{\mathbb{CP}^2}$

Theorem [McDuff] every (V, C, ω) covers a minimal $(\bar{V}, \bar{C}, \bar{\omega})$, obtained by blowing down a finite collection of exceptionnal curves in $V \setminus C$. Given these, $\bar{\omega}$ is unique up to isotopy.

Upshot: the classification problem boils down to the minimal case.

2) Main tool: the adjunction formula. Let S be a rational embedded holomorphic curve. Then $TV|_S = TS \oplus \nu S$

$$\Rightarrow c_1(TV|_S) = c_1(TS) + c_1(\nu S)$$

$$\Rightarrow c_1(V)\cdot[S] = \chi(S) + [S]\cdot[S] \quad \leftarrow \text{adjunction formula}$$

Mega lemma: $\exists$ tame J such that $[C]$ may be represented by a J-hol cusp curve $S = S_1 \cup \dots \cup S_m$, where $[S_i] = A_i$ is J-indecomposable, and J is regular for all A_i-curves. Moreover, the S_i's are distinct embedded curves with self-intersection -1, 0 or 1, and there is at least one component with $A_i \cdot A_i \geq 0$.

Rem. J-indecomposable $\Rightarrow$ moduli space of curves in class A is compact.

Rem: embedded $\Rightarrow$ can use adjunction formula.

Proposition 1: if F is a rational B-curve in (V,ω) where B is simple and $B\cdot B = 0$ then $\exists\, \pi: V \to M$ compatible with ω (ie ω is non-degenerate on the fibers), with F a fibre.

Proof: pick J a tame a-$\mathbb{C}$-str. which splits near F. If f is a J-hol. parametrization of F, then $c_1(\nu F) = 0$ (since J splits).

Automatic transversality $\Rightarrow (f, J)$ is regular.

B simple $\Rightarrow \mathcal{M}_{0,1}(J,B)$ is compact! (we just keep track of 1 marked point; no condition on it)

(0: genus, 1: #marked points)

The dimension is $\underbrace{2n}_{g=0} + 2c_1^{(c)} + \underbrace{2k}_{\text{marked pts}} \underbrace{- 6}_{\dim PSL(2;\mathbb{C})}$

By adjunction, $c_1(C) = \chi(S) + [S]\cdot[S] = 2 + 0 = 2$. So, our dimension is $4 + 4 + 2 - 6 = 4$.

Now, consider $ev: \mathcal{M}_{0,1}(J,B) \to V$, map between 4-dim spaces. Since $B\cdot B = 0$, there is at most one B-curve through each point in V (by positivity of intersection). We already have the curve F. $\Rightarrow \deg(ev) \leq 1$.

In a nbhd of F, there is a family of such J-hol curves, so the degree of ev is ≥ 1. Rem: 2-dim family, without the marked points.

$\Rightarrow$ Degree is exactly 1. So there is exactly 1 J-hol B-curve through every point of V. These form the fibers of a cont. surjection $\pi: V \to M$.

Fibers are holomorphic $\underset{\text{tame}}{\Rightarrow}$ symplectic, so π is compatible with ω. $\square$

Rem: "B is simple" is very often satisfied: any class B with $B.B = 0$ in V minimal is J-simple for almost every J.

Rem: ω compatible with π $\Rightarrow$ ω determined up to isotopy by $[\omega] \in H^2(V)$

Proposition 2: (V, C, ω) minimal, $C \cdot C \geq 0$. Then, (in notations of the mega lemma)

(1) If $S_i \cdot S_i = 1$ for some i, then $V \cong \mathbb{CP}^2$ and $m = 1$ or 2.

(2) If $S_i \cdot S_i = -1$ for some i, then $V \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and there is only one such i, and all other S_i's are homologous with $S_j \cdot S_j = 0$.

(3) If $S_i \cdot S_i = 0$ $\forall i$, then all S_i's (except maybe 1) are homologous, and V is an S^2-bundle (the base of S^2-bundle)

Comments on proof:

(1) Consider $ev: \mathcal{M}_{0,2}(J, A) \to V \times V$; it has degree 1, so there is a unique J-hol A-curve through each pair of points.


$\rightsquigarrow$ construct $\mathbb{CP}^2$ using these lines through p that fill in the whole manifold.

(2) Adjunction $\Rightarrow$ contradiction. (3) also. " $\square$ "

Propositions 1 & 2 $\Rightarrow$ diffeo type of V, in McDuff's theorem. The symplectomorphism type determined just by $[\omega] \in H^2$ up to isotopy.

Corollary: diffeo type determined by $p = C \cdot C \geq 0$ for $p \neq 0$ or 4.

$p = 0 \Rightarrow S^2$-bundle, by proposition 1 (but we don't know over what)

$p = 1 \Rightarrow \mathbb{CP}^2$ by Gromov, and C is sent to a complex line.

$p \geq 2$: $p = C \cdot C = (A_1 + \dots + A_m)^2 = \sum A_i^2 + \sum_{i<j} 2 A_i \cdot A_j$. So

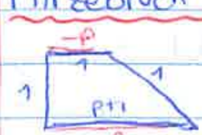
in case (2) above: $p = -1 + \text{even} = \text{odd}$

(3): $p = 0 + \text{even} = \text{even}$

So, p odd $\Rightarrow V \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (the non-trivial S^2-bundle over S^2)

$p \geq 2$ even $\Rightarrow S^2 \times S^2$ (uses ≥ 2 ?: there must be a section amongst the S_i's, by the "except maybe 1" before

Hirzebruch surfaces: $\mathbb{F}_p \cong \mathbb{F}_{p+2}$ $\forall p$, but no 2 are symplectomorphic.



ex: $\mathbb{F}_0 = (S^2 \times S^2, \omega_{std})$ $\quad \mathbb{F}_1 = (\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \omega)$

$\mathbb{F}_2 = (S^2 \times S^2, \text{other } \omega)$

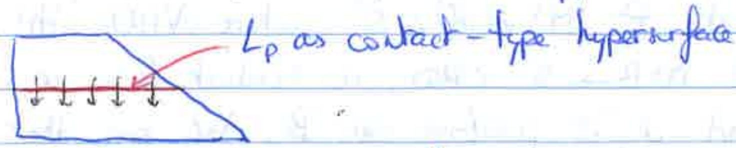
self-intersection 4: 

$p = 4$: $(S_1 + S_2)^2 = 4$, can get $(\mathbb{CP}^2, \text{quadric})$ also.

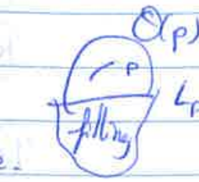
3) We can get $\mathbb{F}_p$ by gluing $\mathcal{O}(p)$ to $\mathcal{O}(-p)$ (or rather, the disk bundles).

<u>Claim:</u> $\partial(\mathcal{O}(p)) = L_p = L(p,1)$

$\partial(\mathcal{O}(-p)) = L_p$ with opposite orientation.



We can see that using Heegaard decomposition.



Punchline: suppose we want to classify fillings of L_p.
Stick on a cap $\mathcal{O}(p)$; this will contain a (-1)-curve.
By McDuff, minimal such compact manifolds are determined uniquely up to symplectomorphism if we fix $[\omega]$, and up to diffeo for $p \neq 0$ or 4.

$\Rightarrow L_p$ have minimal symplectic fillings.
For $p \neq 4$, these are unique up to diffeomorphism.
If we fix $[\omega]$, unique up to symplectomorphism.
(and L_4 has exactly 2 non diffeomorphic fillings)

⚠ McDuff classifies pairs (V, C). It is important to know where C goes; for example, it should not go through the L_p, in the example above.