Mapping Class Group Factorizations as Lefschetz fibration fillings.

- Bahar

Goal: study how the topology of the total space of an exact Lefschetz fibration (LF) ~~[illegible] topology~~ is described by using a distinguished basis of vanishing paths.

Main ingredients: Lefschetz fibration. & OPEN BOOK decompositions.

Lefschetz fibration:

$$f: \begin{tikzcd} W^4 \arrow[d] \\ D^2 \end{tikzcd}$$

smooth fibration

(a) critical points $\text{Crit}(f)$ = isolated nondegenerate in $\text{Int}(W)$.

(b) local behavior around critical points

$$(z_1, z_2) \longmapsto z_1^2 + z_2^2$$

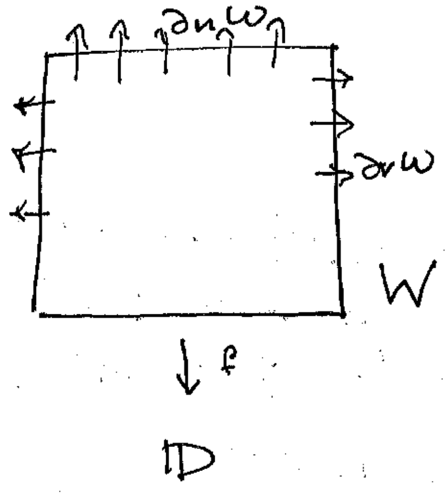
(W^4, ω)
↑ symplectic form.

$F = f^{-1}(a)$ a = regular value

$$\omega = \underset{\text{Giroux or thurston}}{\omega_F} + \underbrace{K}_{\text{constant}} \cdot f^*(d\sigma)$$

$d\sigma$ = canonical symplectic form on $\mathbb{D}^2$

Total space



$f^{-1}(\partial \mathbb{D}^2) = \partial_v W$ ~~horizontal~~ vertical

$\bigsqcup_{z \in \mathbb{D}} \partial F_z = \partial_h W$ horizontal
↓
nbd of the binding.

The main motivation to work w/ LF is that they provide a manageable approach to study symplectic 4 manifolds.

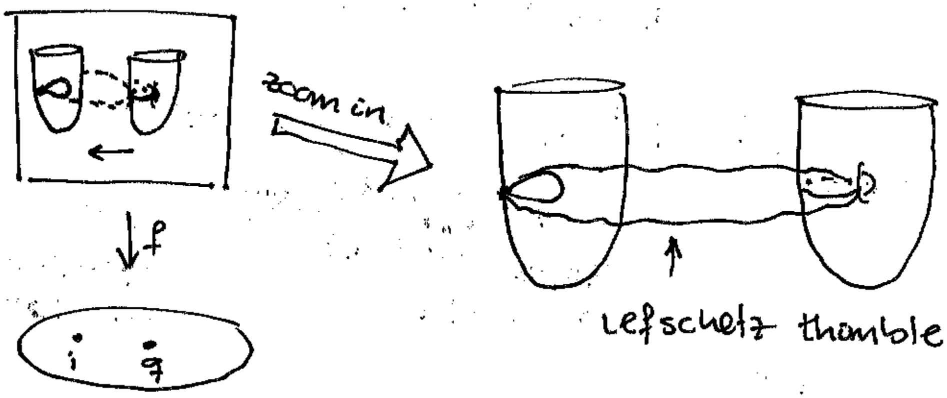
Fix a point $q_* \in \mathbb{D} \setminus \text{crit}(f)$ and let $F = f^{-1}(q_*)$ be the corresponding fiber. Then we can consider

$$\Psi : \pi_1 (\mathbb{D} \setminus \mathrm{Crit}(f), q_*) \longrightarrow \pi_1 (\mathrm{Diff}^+(f))$$

$$\gamma \longmapsto \left\{ \begin{array}{l} \text{isotopy class of orientation} \\ \text{preserving diffeomorphisms of} \\ F \text{ fixing } \partial F \text{ given by} \\ \text{parallel transport along } \gamma \end{array} \right\}$$

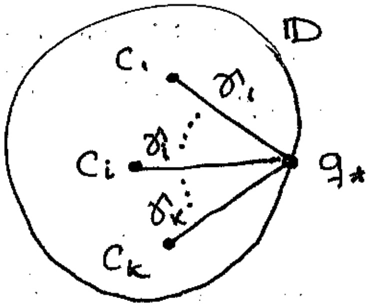
$:\equiv \Psi(\gamma)$ monodromy of the LF f.

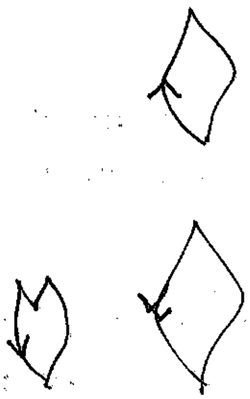
DEFINITION: A vanishing cycle is a homology cycle of F which collapses to the unique singular point on the fiber under the parallel transport along γ.



i: critial value
q: regular value

DEFINITION: A distinguished basis of vanishing paths ~~an ordered~~ $\{\gamma_1, \dots, \gamma_k\}$ is an ordered set of vanishing paths γ_i for each critical values c_i of f starting at q_* and ending at c_i, the critical value





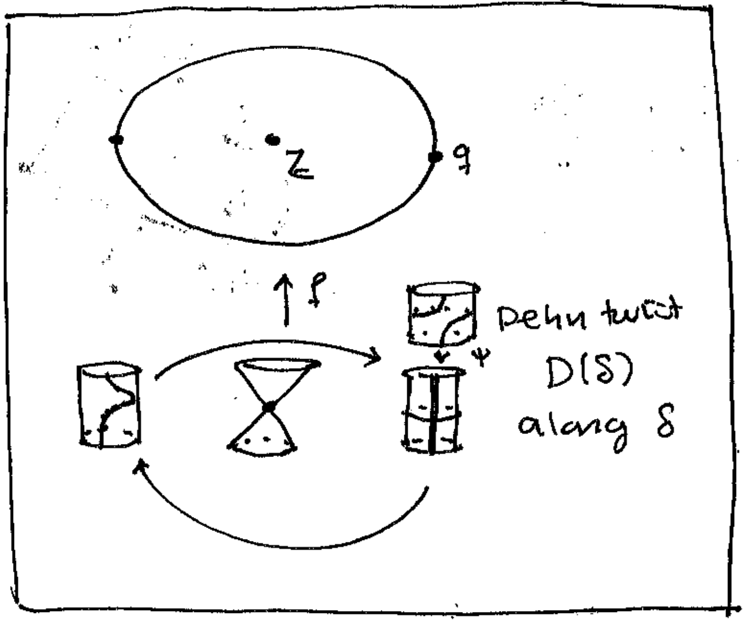
Recall: Given a symp Lefschetz fibration $W^4 \to \mathbb{D}$

$f^{-1}(\partial \mathbb{D}) = F$ bundle over S^1

$\simeq F \times [0,1] / (1,x) \sim (0, \psi(x)) \equiv$ mapping torus of OBD of ∂W

where ψ = geometric monodromy

Moreover, the monodromy of a LF is a Dehn twist. (positive).



$$\psi = D(\delta_1) \cdots D(\delta_k) \in \mathrm{Map}(F)$$
$$\| $$
$$\pi_0(\mathrm{Diff}^+(F, \partial F))$$

critical values $c_1, \dots, c_n$ w/ distinguing basis of vanishing paths $\gamma_1, \dots, \gamma_k$ & $\delta_1, \dots, \delta_k$ vanishing cycles above $\gamma_1, \dots, \gamma_k$.

This is a monodromy factorization.

To every LF, we associate a monodromy factorization in the Map(F)

<u>Fact</u>: The monodromy factorization depends on

(1) topology of the fiber F

(2) the choice of the vanishing paths.

Let $(\gamma_1', \dots, \gamma_K')$ be a distinct basis of vanishing paths of f.

Then it induces a different monodromy factorization

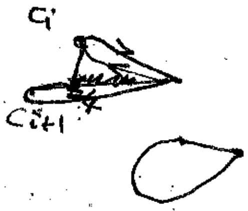
<u>Q</u> How are they related?

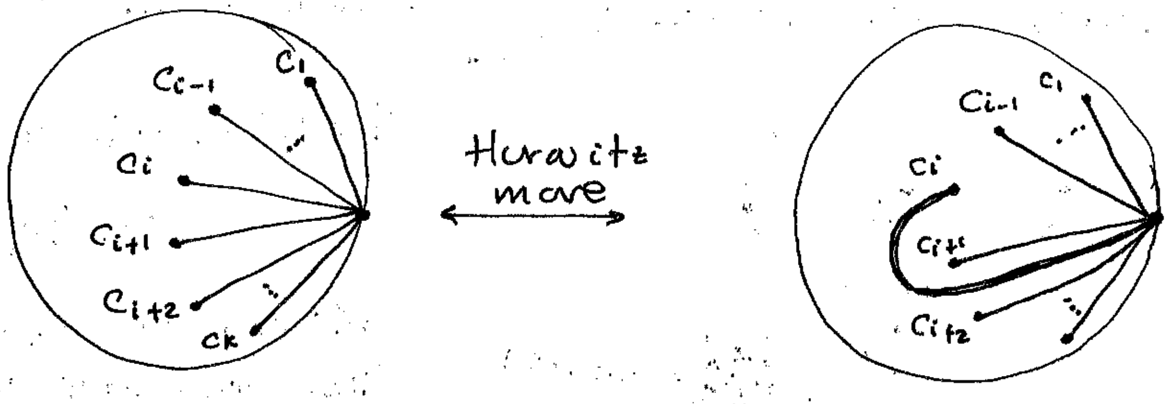
<u>A</u>: By Hurwitz moves

$$\begin{tikzcd}(\delta_1, \dots, \delta_{i-1}, \delta_i, \delta_{i+1}, \dots, \delta_k) \arrow[d, "\text{Hurwitz}"] \\ (\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, D(\delta_{i+1})(\delta_i), \delta_{i+2}, \dots, \delta_k)\end{tikzcd}$$

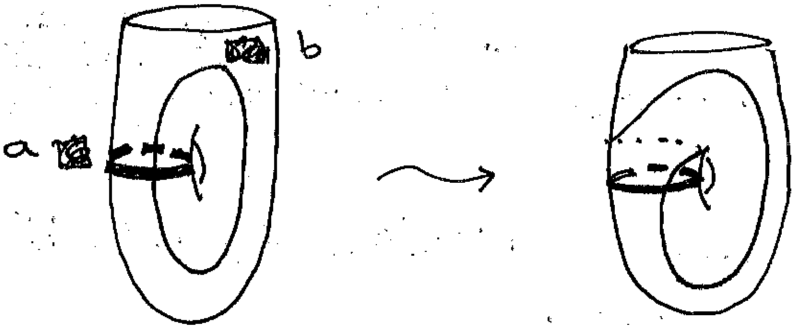
vanishing cycle level

Now on the vanishing path level:





γ_i twisting around γ_{i+1}



Dehn twist

$D(\delta_i) = \tau_i$

$\tau_a^{-1} \tau_b \tau_a(b) = \tau_{\tau_a(b)}$

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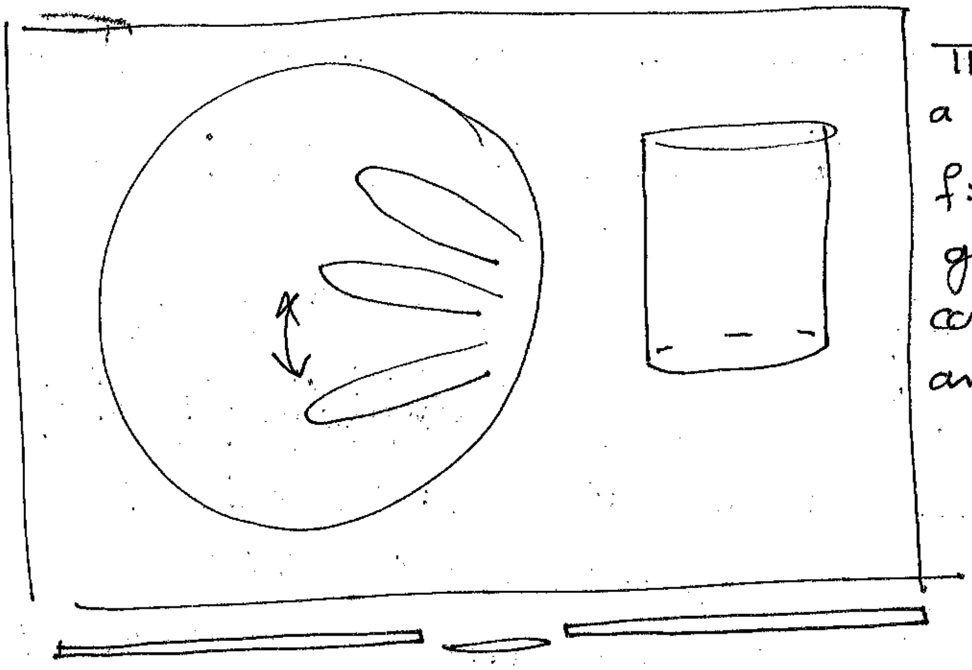
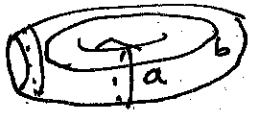
vanishing paths = legendrian circles = ∂ of lagrangian disk

Classification of Lefschetz fibrations over $\mathbb{D}$ amounts to the classification of monodromy factorizations in Map(F) up to Hurwitz equivalence.



THEOREM: Let F be a regular fiber of LF $f: W^4 \to \mathbb{D}$ with genus g and boundary components.
and if $2-2g-n<0$ then,

$$\left\{\begin{array}{l}\text{factorizations of}\\ \text{boundary Dehn}\\ \text{twists as a product}\\ \text{of right handed Dehn}\\ \text{twists}\end{array}\right\} \Big\updownarrow \text{1-1} \; \left\{\begin{array}{l}\text{(w/o) } \partial \text{ components}\\ \text{genus } g \text{ LF}\\ \text{over } \mathbb{D}\end{array}\right\} \Big/ \text{isotopy}$$

EX $F =$ 

$$\mathrm{Map}(T^2) = SL(2,\mathbb{Z})$$

$$\mathrm{Map}(S) = \widetilde{SL}(2,\mathbb{Z}) \simeq B_3$$

$$= \langle \tau_a, \tau_b \mid \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b \rangle$$

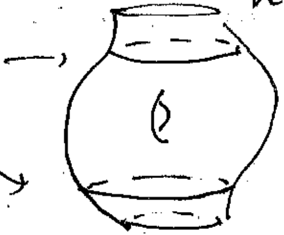
$(g=1 \quad n=1)$

$\tau_c = (\tau_a \; \tau_b)$ central element
generates the kernel of $\mathrm{Map}(F) \to SL(2,\mathbb{Z})$

Factorization of bdy Dehn twist genus g, $n = \#$ boundary components. $\longleftrightarrow$ Genus g Lefschetz fillings of corresponding $OBD_{g,n}$

$\Big\updownarrow$ Giroux

$(Y_{g,n}, \xi_{g,n})$



Dehn twist on these curves →
order doesn't matter

$S^1 \hookrightarrow Y_{g,n}$ where Euler class $-n$

$\downarrow$

Σ_g (closed)

THM (Auroux) The monodromy factorizations

$$\psi: (\tau_a^{-3}\tau_b\tau_a^{3})(\tau_b)(\tau_a^{-3}\tau_b\tau_a^{-3})(\tau_a) ,$$

&

$$\psi: (\tau_a^{-2}\tau_b\tau_a^{2})(\tau_b)(\tau_b)(\tau_a^{2}\tau_b\tau_a^{-2})$$

of ψ on $Map(F) = \widetilde{SL}(2,\mathbb{Z})$ (OJO: they are distinct factorization of the same monodromy) define inequivalent genus 1 LF's $f_1: W_1 \to \mathbb{D}$ & $f_2: W_2 \to \mathbb{D}$. The corresponding Stein filling W_1, W_2 of the OB w/ monodromy are distinguished by their ~~ma~~ first homology $H_1(W_1) = 0$
$H_2(W_2) = \mathbb{Z}_2$

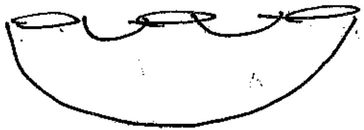
Note $H_1(W_i) = H_1(F_i) \big/$ span of vanishing cycles.

- Given a LF w/ fiber $g=1$ $n=3$ you can find ∞ many stein fillings for the corresponding $OBD(F,\psi) = M^3$

$F=$

distinguished by their 1st homology classes. (Baykur - Stipsicz)

- planar surface $g=0$, planar OBD if F has $g=0$



Lens space $(L(p,q), \xi)$ ↑specific

has finite # of stein fillings.

Open Question:

~~Example~~ Every filling from a planar open book comes from a factorization. For $g=0$ & there should only be finitely many fillings.