Bahar - Mapping class group factorizations as Lefschetz fibration fillings

<u>Goal</u>: to study how the topology of the total space of an exact Lefschetz fibration (LF) is described by using a distinguished basis of vanishing paths.

<u>Main ingredients</u>: LF and OBD's monodromy of the boundary of the total space.

<u>Lefschetz fibration</u>: $f: W^4 \to \mathbb{D}^2$ such that

(a) $\mathrm{crit}(f)$ are isolated non-degenerate in $\mathrm{int}(W)$.

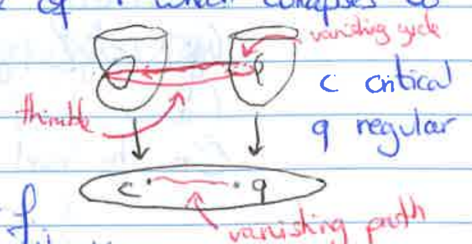
(b) local behaviour around critical points: $(z_1, z_2) \mapsto z_1^2 + z_2^2$

(c) for $F = f^{-1}(q)$ for q regular value, $\omega = \omega_F + K \cdot f^*(d\sigma)$, for K large. canonical sympl. form on $\mathbb{D}^2$

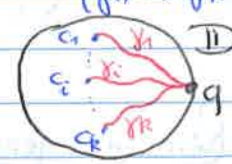
Recall: $\partial_v W = f^{-1}(\partial \mathbb{D}^2)$ and $\partial_h W = \coprod_{z \in \mathbb{D}^2} \partial F_z$.

<u>Motivation</u>: they provide some manageable approach to studying symplectic 4-mflds.
Fix a point $q_* \in \mathbb{D}^2 \setminus \mathrm{crit}(f)$, and let $F = f^{-1}(q_*)$ be the corresponding fiber. Then, we can consider $\Psi: \pi_1(\mathbb{D} \setminus \mathrm{crit}(f), q_*) \to \pi_0(\mathrm{Diff}^+(F))$.
$\gamma \mapsto \{$isotopy classes of diffeos of F fixing ∂F given by parallel transport along $\gamma\}$.
It is the monodromy of the LF f.

<u>Definition</u>: a <u>vanishing cycle</u> is a homology cycle of F which collapses to the unique singular point of the critical fiber under the parallel transport. The corresponding <u>thimble</u> is the image of the entire parallel transport.
The <u>vanishing path</u> is the image of the thimble under f.
Rem: really, choose a path first, and then get a thimble.



<u>Definition</u>: a distinguished basis of vanishing paths $\{\gamma_1, \dots, \gamma_k\}$ is an ordered set of vanishing paths, for each critical value c_i of f, starting at q ($= q_*$ above) and ending at the critical value c_i.

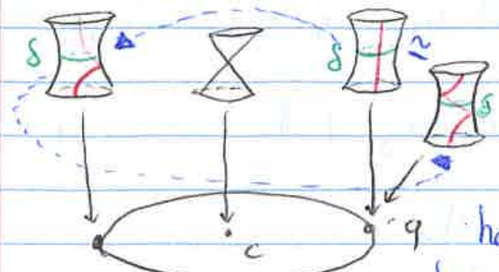


Rem: nowadays, we should think of the vanishing cycle as a Lagrangian, instead of a homology class. That's what symplectic geometry buys us.

<u>Recall</u>: given a symplectic LF: $W \to \mathbb{D}^2$, $f^{-1}(\partial W)$ is a F-bundle over S^1 $\simeq F \times [0,1] / (1,x) \sim (0, \Psi(x))$ ($\equiv$ mapping torus of OBD of ∂W), Ψ is the

geometric monodromy. Moreover, the monodromy around one critical point of a LF is a (right-handed) Dehn twist.

$D(\delta) :=$ Dehn twist along δ.



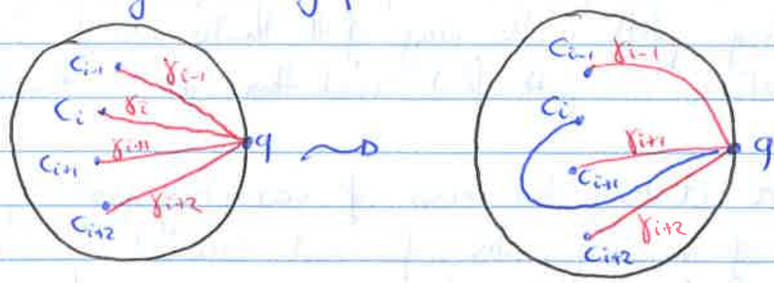
For critical values $c_1, \dots, c_n$ along with a distinguished basis of vanishing paths $\gamma_1, \dots, \gamma_k$, have $\psi = D(\delta_1) \cdots D(\delta_k) \in \mathrm{Map}\, F = \pi_0(\mathrm{Diff}^+(F, \partial F))$, where $\delta_i :=$ vanishing cycle over γ_i.

To every LF, we can associate a monodromy factorization in $\mathrm{Map}(F)$.
Fact: the monodromy factorization depends on
(1) the topology of the fiber F.
(2) the choice of the vanishing paths.

Let $(\gamma_1', \dots, \gamma_k')$ be another distinguished basis of vanishing paths of f. Then, it induces a different monodromy factorization.
Q: How are they related?
A: Hurwitz moves: every 2 dist. bases of van. paths are related via elementary Hurwitz moves: on vanishing cycles, it looks like:

$$(\delta_1, \dots, \delta_{i-1}, \delta_i, \delta_{i+1}, \dots, \delta_n) \mapsto (\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, D(\delta_{i+1})(\delta_i), \delta_{i+2}, \dots, \delta_n)$$

On the level of vanishing paths.

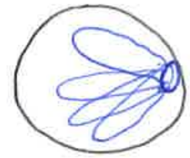


Rem. in general, $f^{-1} \tau_b f = \tau_{f(b)}$. (*)

For monodromies: $D(\delta_1) \dots D(\delta_k)$

$$D(\delta_1) \dots D(\delta_{i-1}) \overbrace{D(\delta_{i+1})\, D^{-1}(\delta_{i+1})}^{id} \underbrace{D(\delta_i)\, D(\delta_{i+1})}_{D(D(\delta_{i+1})(\delta_i)) \text{ by } (*)} \dots D(\delta_k)$$

we see they are equal.



If we think of the thimbles as the cores of the handles we attach, this move corresponds to a handle slide. This corresponds as summing the generators corresponding to these 2 handles on homology, which is what a Dehn twist does.

Note: the classification of LF over the disk amounts to that of monodromy factorization in Map(F) up to Hurwitz equivalence.

Theorem: let F be a regular fiber of a LF $f: W^4 \to \mathbb{D}$ with genus g and n boundary components. If $2-2g-n<0$, then

$$\left\{\begin{array}{l}\text{factorization of boundary Dehn}\\ \text{twists as a product of right-handed}\\ \text{Dehn twists}\end{array}\right\} \Big/ \text{Hurwitz moves} \xleftrightarrow{1-1} \left\{\begin{array}{l}\text{genus } g \text{ with } n\ \partial \text{ components}\\ \text{LF over } \mathbb{D}\end{array}\right\} \Big/ \text{isotopy}$$

ex: F =  $(g=1, n=1)$. We know $Map(T^2) = SL(2;\mathbb{Z})$. It turns out that $Map(F) = \widetilde{SL}(2;\mathbb{Z}) \simeq B_3 = \langle \tau_a, \tau_b \mid \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \rangle$.
$\tau_c = (\tau_a\tau_b)^6$ central element, generates the kernel of $Map(F) \to SL(2;\mathbb{Z})$.

Laura: Factorization of ∂ Dehn twists (g,n) $\longleftrightarrow$ Genus g Lefschetz fillings of corresponding $OBD_{g,n}$

$\updownarrow$ [Giroux]

$(Y_{g,n}, \xi_{g,n})$

where $S^1 \hookrightarrow Y_{g,n} \to \Sigma_g$ is the Euler class $=n$ circle bundle.

Theorem [Auroux] the monodromy factorizations 
$\Psi: (\tau_a^{-3}\tau_b\tau_a^{3})(\tau_b)(\tau_a^{-3}\tau_b\tau_a^{-3})(\tau_a)$ and
$\Psi: (\tau_a^{-2}\tau_b\tau_a^{2})(\tau_b)(\tau_b)(\tau_a^{2}\tau_b\tau_a^{-2})$ of Ψ in $\widetilde{SL}(2,\mathbb{Z}) = Map(F)$
define inequivalent genus 1 LF's $f_i: W_i \to \mathbb{D}$. The corresponding Stein filling W_1, W_2 of the OB with monodromy Ψ are distinguished by their first homology: $H_1(W_1)=0$, $H_1(W_2)=\mathbb{Z}_2$.

Rem: $H_1(W_i) = H_1(F_i) \big/ \text{span of vanishing cycles}$

ex: $M^3 = OB(F, \psi)$

$\hookrightarrow$ 

has infinitely many Stein fillings distinguished by their 1^{st} homology classes. [Baykur - Stipsicz)

Rem: # Stein fillings of Lens spaces : $|(L(p,q), \xi_{p,q})| < \infty$.
[Eliashberg - Lisca - McDuff]