

Lefschetz fibration + open books

everything will be for 4 & 3 dim mflds but can be generalized.

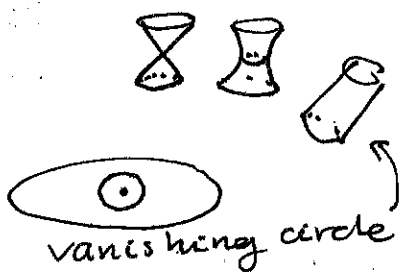
DEFINITION: A Lefschetz fibration on a manifold M is a map

$\pi: M \rightarrow D^2$, such that π has finitely many critical values $t_1, \dots, t_n \in \mathring{D} \nsubseteq \exists$ unique ^{critical point} $p_i \in \pi^{-1}(t_i)$, $\nsubseteq \exists$ local coords such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.

$M^4 \simeq F \times D^2$ away from the critical values.

$$\pi(x_1 + iy_1, x_2 + iy_2) = \underbrace{x_1^2 + x_2^2 - y_1^2 - y_2^2}_{c} + i \underbrace{2(x_1 y_1 + x_2 y_2)}_{0}$$

$$\pi^{-1}(c) \cap U = \{ (z_1, z_2) \mid c \neq 0 \}$$



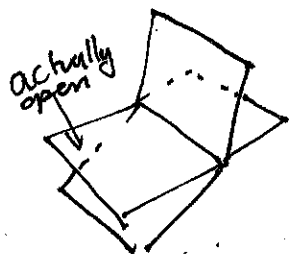
DEFN:

Vanishing cycle \nsubseteq origin form a thimble

Q where is the 2 handle?
no to see

$$-f = -(x_1^2 + x_2^2 - y_1^2 - y_2^2)$$

OPEN BOOK decomposition of M is a pair (B, T)



where

- (1) B is an oriented link "binding"
- (2) $\pi: M \setminus B \rightarrow S^1$ is a fibration s.t. $\pi^{-1}(\theta)$ is the interior of compact surface Σ_θ $\nsubseteq \partial \Sigma_\theta = B$ "pages"

$\theta \in S^1$

DEFINITION: An abstract open book is a pair (Σ, ϕ) s.t.

① Σ is an oriented compact surface

② $\phi: \Sigma \rightarrow \Sigma$ is a diffeo that is the identity on a nbhd of $\partial \Sigma$

ϕ is called monochromy

REMARK: Can build a 3-mfld from an

$$M_\phi = \sum_\phi \bigcup_{\psi} \bigsqcup_{|DZ|} S^1 \times D^2$$

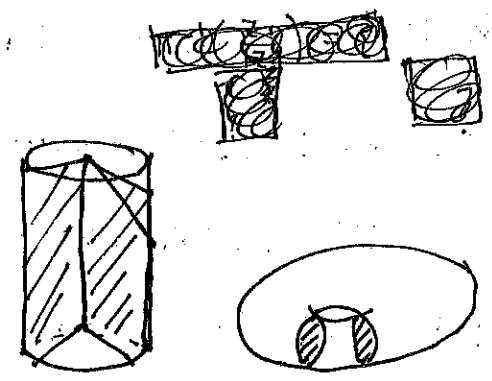
↑
of boundary components.

E.g. $S^3 \subset \mathbb{C}^2$

$$U = \{z_1 = 0\}$$

$$\pi_U : S^3 / U \rightarrow S^1$$

$$(z_1, z_2, \sqrt{|z_1|^2 + |z_2|^2}) \mapsto \frac{z_1}{|z_1|}$$

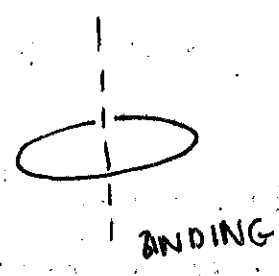


Binding = S^1
pages = shaded regions

E.g. $U = \text{Hopf link}$

$$U = \{(z_1, z_2) \mid z_1 z_2 = 0\}$$

$$(z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$$

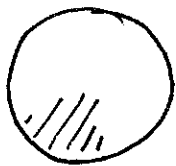
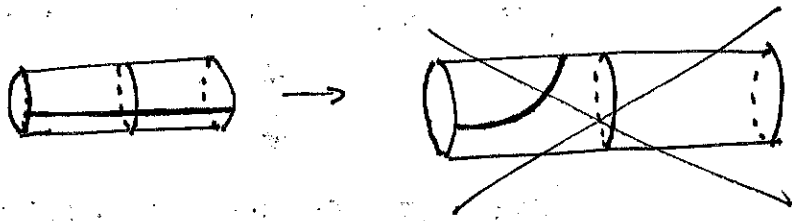


THM: Every closed oriented 3-mfld has an Open Book decomposition (OBD).

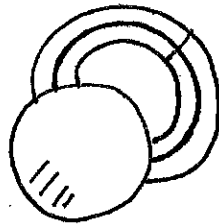
THM (Giroux) M is a closed oriented 3-mfld, then \exists

bijection $\{ \text{oriented contact structures} \} / \text{isotopy} \leftrightarrow \{ \text{OBD} \} / \text{stabilization}$

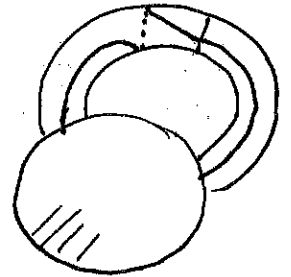
stabilization = 1 handle & then applying a Dehn twist. 20/May/17 3



Disk = page



attach a 1-handle
 $D^1 \times D^1$



Dehn twist
around core of
1-handle

$\pi: M \rightarrow D^2$ V_1 = vertical boundary correspond to \mathbb{F} over ∂D

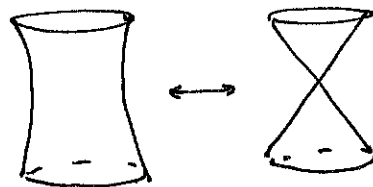
V_2 = horizontal ∂ : union of ∂ fibers.

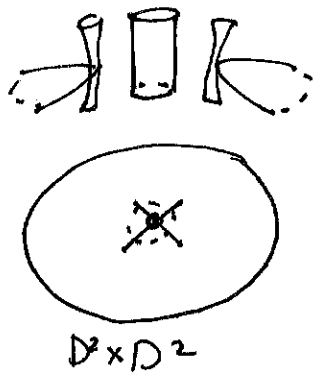
Every $x \in \partial M$ lies in either V_1 or V_2 .

Fibers of the vertical boundary = pages
horizontal boundary = binding

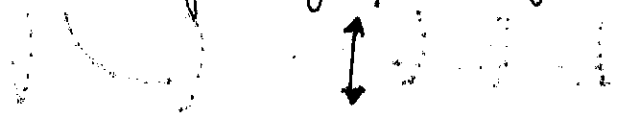
For every critical point, attach a 2-handle along a vanishing cycle. In the boundary apply Dehn surgery along the vanishing cycle.

Contact structure?





Stein filling up to deformation



Lefschetz fibrations/stabilization

Fix contact mfd & look at OBD & factorizations into right handed Dehn twists.

