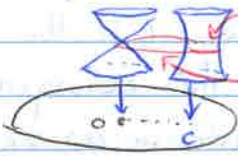
Alvin - Lefschetz fibrations and open books

Definition: a Lefschetz fibration on a 4-manifold M is a map $\pi : M \to D^2$ such that
- π has finitely many critical values $t_1, \dots, t_n \in \mathring{D}_n$
- $\exists$ unique critical point $p_i \in \pi^{-1}(t_i)$
- $\exists$ local coordinates near p_i such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Rem: away from these critical values, π is actually a fibration, with fiber F.

Take $\mathcal{U}$ a nbhd of p_i, in which $\pi(x_1+iy_1, x_2+iy_2) = x_1^2+x_2^2-y_1^2-y_2^2+2i(x_1y_1+x_2y_2)$.

For c real, $\pi^{-1}(c) \cap \mathcal{U} = \{(z_1, z_2) \mid x_1^2+x_2^2-y_1^2-y_2^2 = c,\ x_1y_1+x_2y_2 = 0\}$



vanishing cycle : imaginary part of coordinates is 0.

thimble = origin + vanishing cycles

From critical fiber to other ones, we attach a 2-handle. $f = -x_1^2 - x_2^2 + y_1^2 + y_2^2$.

Definition: an open book decomposition of M is a pair (B, π) where
(1) B is an oriented link (the "binding")
(2) $\pi : M \setminus B \to S^1$ is a fibration, such that $\pi^{-1}(\theta)$ is the interior of a compact surface Σ_θ (the "page"), and $\partial \Sigma_\theta = B$.

Definition: an abstract open book is a pair (Σ, ϕ) such that
(1) Σ is an oriented compact surface
(2) $\phi : \Sigma \to \Sigma$ is a diffeo which is the identity in a nbhd of the boundary; it is called the "monodromy"

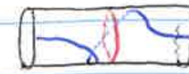
Rem: can build a 3-mfld : $M_\phi = \underbrace{\Sigma_\phi}_{\text{mapping torus}} \cup_{id} \bigsqcup_{|\partial\Sigma|} S^1 \times D^2$

Example: $S^3 \subseteq \mathbb{C}^2$: $\mathcal{U} = \{z_2 = 0\}$, $\pi_\mathcal{U} : S^3 \setminus \mathcal{U} \to S^1 : (z_1, z_2) \mapsto \frac{z_1}{|z_1|}$
$\hookrightarrow$ ~~bound~~ binding $= S^1$, pages = shaded region.

Example: $\mathcal{U}$ = Hopf link = $\{(z_1, z_2) \mid z_1 z_2 = 0\}$; $\pi : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$

Theorem: every closed oriented 3-mfld has an OBD.

Theorem [Giroux] if M is closed oriented 3-mfld, then $\exists$ bijection
$$\{\text{oriented contact structures}\}/\text{isotopy} \longleftrightarrow \{\text{OBD}\}/\text{stabilization}$$



Stabilization: attach 1-handle, and compose monodromy with right-handed Dehn twist.

Now, consider a Lefschetz fibration $\pi : M \to D^2$; it has
* a vertical boundary : corresponds to F over ∂D
* a horizontal boundary : union of boundary of fibers

Every $x \in \partial M$ lies in one of these 2. This gives an open book: The fibers of the vertical boundary are the pages, and the horizontal boundary is a nbhd of the binding.

Rem: for every critical point, attach a 2-handle along a vanishing cycle. In ∂: ~~vanishing~~ Dehn surgery along vanishing cycle.

Clearer: start with regular point; a nbhd consists of regular fibers. For each crit value of π, attach a 2-handle, corresponding to the thimble for that critical value. On the boundary, the effect of this is a Dehn twist along the corresponding vanishing cycle: this Dehn twist is the new monodromy as we go along $S^1 = \partial D^2$.

$\{\text{Stein fillings}\}/\text{deformation} \longleftrightarrow \{\text{Lefschetz fibrations}\}/\text{stabilization}$.

Fix contact manifold; look at OBD and factor the monodromy into right-handed Dehn twists.