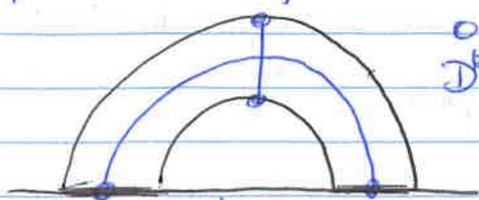
Orsola – Kirby calculus for Stein manifolds

Ref: Gompf's paper and book.

An n-dim'l k-handle is $D^k \times D^{n-k}$ attached to the boundary of an n-manifold X, along $S^{k-1} \times D^{n-k}$.



$0 \times D^{n-k}$ = cocore

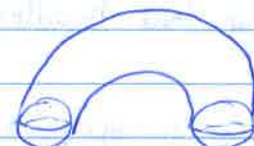
$D^k \times 0$ = core

$S^{k-1} \times D^{n-k}$ = attaching region

$S^{k-1} \times 0$ = attaching sphere

2-dim 1-handle: 

4D: 0-handle: 

1-handle $D^1 \times D^3$:  $S^0 \times D^3$

notation: 

2-handle: $D^2 \times D^2$: along a circle. If the circle goes through the 1-handle: get 

along $S^1 \times D^2$

$f: S^1 \times D^2 \longrightarrow S^1 \times D^2$ is the framing

$\mu \longmapsto p\mu +$ [illegible] λ ; $r := p$ [illegible].

3D: 0-handle: B^3

1-handle: $B^1 \times B^2$, along $S^0 \times B^2$: 

2-handle: $B^2 \times B^1$, along $S^1 \times B^1$:  thick capping.

framings = homeo of annulus $S^1 \times B^1$; there are only 2 of them.

3-handle: B^3

4D: 3-handle: $B^3 \times B^1$, attached along $S^2 \times B^1$

4-handle: B^4

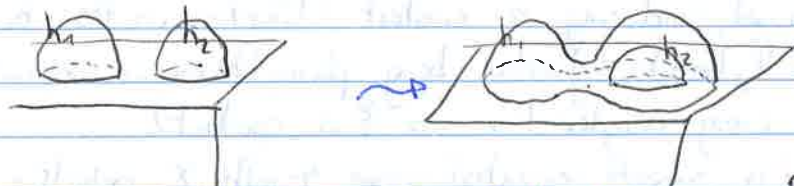
If $\phi(S^{k-1})$ is the attaching sphere, then a framing is $f : \nu\,\phi(S^{k-1}) \longrightarrow S^{k-1} \times \mathbb{R}^{n-k}$.

In 4D, we get what's called a Kirby diagram. We'd like to do moves on it, to simplify it.

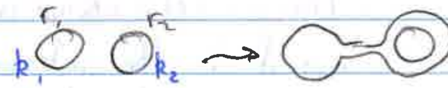
1) Handle sliding: h_1, h_2 handles of index k. Take the attaching sphere of h_1, and slide it over the other one until it's back on the original manifold.

$\left.\begin{matrix} k=1 \\ n=2 \end{matrix}\right\}$:

$\left.\begin{matrix} k=2 \\ n=3 \end{matrix}\right\}$.



The attaching circles look like :



$r_1' = r_1 + r_2 + 2\,lk(k_1, k_2)$

In homology: if these 2 handles give $\alpha_1, \alpha_2 \in H_2(X)$, this move gives ~~$(\alpha_1, \alpha_2) \mapsto$~~ $\alpha_1 \mapsto \alpha_1 + \alpha_2$.

Rem: the intersection in homology corresponds to the linking number of the attaching spheres.

2) Handle cancellation: h_1 index k, h_2 index $k-1$

The belt sphere of h_1 and the attaching sphere of h_2 intersect once.

n=2:

n=4: a 1-handle and a 2-handle cancel out if

= nothing.

part of attaching circle of 2-handle, the other part is in the 1-handle from ⊖ to ⊖.

ex.

: slide the 3 2-handles across the 2-handle, and then simplify:

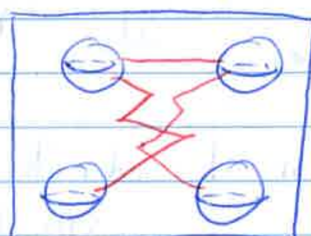
cancel 1 handle with h_1

Rem: we should include the framing for the knots we drawn (which are the attaching circles of the 2-handles). For the Legendrian case, that framing will be canonical.

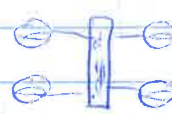
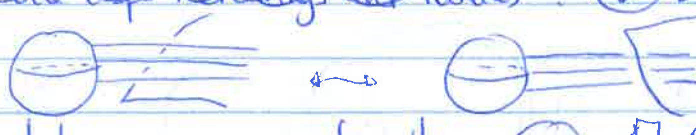
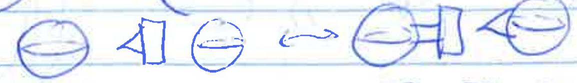
Stein manifolds.

These are the complex manifolds admitting a proper holomorphic embedding into some $\mathbb{C}^N$. Equivalently, it admits an exhausting strictly subharmonic function $f: X \to \mathbb{R}$ ($f^{-1}(c) = \partial f^{-1}((-\infty, c])$.
A Stein domain is a complex compact manifold that admits such an f.
For $\mathbb{C}$-dim $= 2$, these are called Stein surfaces. They have almost complex structures J, inducing a contact structure on the boundary: $\xi = T_p(\partial X) \cap J T_p(\partial X)$ (function being plurisubharmonic means that it is harmonic on every complex line $\Rightarrow \xi$ is contact).

Theorem (Eliashberg) a smooth oriented open 4-mfld X admits a Stein structure $\Leftrightarrow$ it is the interior of some handlebody H such that
(a) every handle has index $k \leq 2$
(b) each 2-handle is attached along a Legendrian knot k_i in ξ on (0-handle $\cup$ 1-handle), and it has framing $tb(k_i) - 1 \in \mathbb{Z}$.

In Kirby diagram:  We need the framing to be $tb(k_i) - 1$.

The canonical framing is $J\tau \in \xi$, it differs by 1 from the product framing.

Theorem [Gompf] K_1 is isotopic to K_2 in "standard form"  if we can go from one to the other by one of 6 moves:
- the Reidemeister moves (3 of them) 
- ~~slide cusps & crossings over handles~~:
- 
- slide cusp over handle: 
- slide crossing over handle: