François-Simmon

CONTACT SURGERY

M^n manifold

$S^k \times D^{n-k} \subset M$

$\Updownarrow$ tubular nhd thm

$S^k \subset M$ with a trivialization of normal bundle

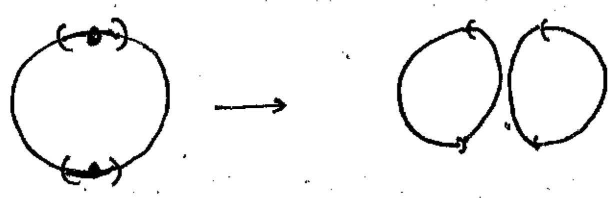
$$\partial(S^k \times D^{n-k}) = S^k \times S^{n-1-k} = \partial(D^{k+1} \times S^{n-1-k})$$

$$M' = (M \setminus \mathrm{int}(S^k \times D^{n-k})) \cup D^{k+1} \times S^{n-1-k}$$

M and M' are cobordant

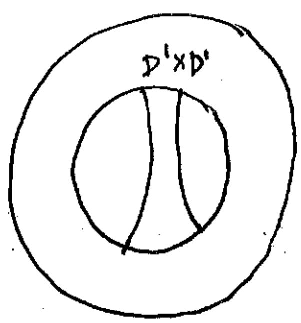
$$W = M \times I \cup D^{k+1} \times D^{n-k}$$

"handle"



$S^0 \times D^1$ $\qquad$ $D^1 \times S^1$

$S^1 \times I$



Contact surgery $(M^{2n+1}, \xi = \ker(\alpha))$

$S^k \subset M$ want that the sphere is isotropic with trivialization of "conformal symplectic normal bundle"

Isotropic submanifolds:

$L \subset (M, \xi)$ isotropic if $TL \subset \xi$

$\xi = \ker(\alpha) \Rightarrow d\alpha|_\xi$ symplectic form on ξ

for $f > 0$ $\quad \xi = \ker(f\alpha) \Rightarrow df\alpha|_\xi = f\, d\alpha|_\xi$

$\Rightarrow \xi$ has a conformal symplectic structure

$\Rightarrow$ complements: $\eta \subset \xi \longrightarrow \eta^\perp \subset \xi$ $\quad \left(\eta^\perp = \{X \in \xi \mid d\alpha(X, Y) = 0 \ \forall\, Y \in \eta\}\right)$

L isotropic means that $TL \subset TL^\perp$

Conformal symplectic normal bundle

$CSN(L) = TL^\perp / TL$ has a conformal symplectic structure induced by ξ.

THEOREM: Suppose $L_i \subset (M_i, \xi_i)$ are isotropic $i = 1, 2$.

$$\begin{array}{ccc} CSN(L_1) & \xrightarrow{\Phi} & CSN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

Φ isometry of conformal symplectic bundle ϕ diffeo

$\Rightarrow \phi$ extends to a contactomorphism.

$$\mathcal{O}_p(L_1) \longrightarrow \mathcal{O}_p(L_2).$$

Proof: $L \subset (M, \xi)$ isotropic

$$TL \subset TL^{\perp} \subset \xi|_L \subset TM|_L$$

$$N(L) \cong TM|_L / \xi|_L \oplus \xi|_L / TL^{\perp} \oplus TL^{\perp}/TL$$

$\|$

$TM|_L / TL$

$$\cong \langle R_\alpha \rangle \oplus T^*L \oplus CSN(L).$$

Recall: Reeb vector field R_α
Pick α s.t. $\xi = \ker(\alpha)$ $\exists$ R_α s.t.

$$d\alpha(R_\alpha, -) = 0$$
$$\alpha(R_\alpha) = 1.$$

$$\xi|_L \twoheadrightarrow T^*L$$
$$Y \longrightarrow d\alpha(Y, -)$$

Kernel $= TL^{\perp}$.

Get $\Psi: NL_1 \xrightarrow{\sim} NL_2$ by taking $R_{\alpha_1} \to R_{\alpha_2}$

$(\phi^*)^{-1}: T^*L_1 \longrightarrow T^*L_2 \Rightarrow$ (tubular nhd) can get an extension

$\tilde{\phi}: \mathcal{O}p(L_1) \longrightarrow \mathcal{O}p(L_2)$ such that

$$D\tilde{\phi}|_{L_1} = D\phi \oplus \Psi$$

$\phi^* \alpha_2 = \alpha_1$ on L_1, do they match elsewhere? $\square$

Yes By Grays theorem

THEOREM: $L_i \subset (M, \alpha_i)$ isotropic $i = 1, 2$

$$\begin{tikzcd} SN(L_1) \arrow[r, "\Phi"] \arrow[d] & SN(L_2) \arrow[d] \\ L_1 \arrow[r, "\phi"] & L_2 \end{tikzcd}$$

Φ is an isometry of symplectic bundles

ϕ diffeo

$\Rightarrow \phi$ extends $\tilde{\phi}: Op(L_1) \longrightarrow Op(L_2)$ $\quad \tilde{\phi}^* \alpha_2 = \alpha_1$

Liaville vecter fields:

V = vecter field on (W, ω) such that

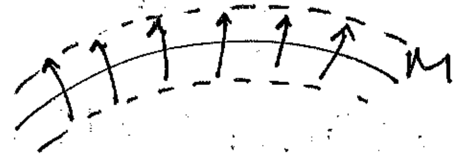
$$\mathcal{L}_V \omega = \omega \iff d\lambda = \omega, \quad \lambda = \omega(V, -).$$

EXAMPLE: $(M, \xi = \ker(\alpha))$ symplectization is

$$\big(\mathbb{R}_t \times M, d(e^t \alpha)\big) \qquad \frac{\partial}{\partial t} \text{ Liaville vecter field here.}$$

$M^{2n-1} \subset (W^{2n}, \omega)$ $\quad V \pitchfork M$ "contact type" $\overset{\text{Liaville}}{\text{vecter field}}$

$\alpha = \lambda|_M$ contact form on M,



Flow of V

$$Op(M) \underset{\text{symplectomorphic}}{\simeq} \big((-\epsilon, \epsilon) \times M, d(e^t \alpha)\big)$$

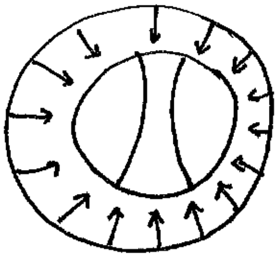
THEOREM: $L_i \underset{\text{isotropic}}{\subset} M_i^{2n-1} \subset (W_i^{2n}, \omega_i)$ $\quad i = 1, 2$

V_i Liaville, $V_i \pitchfork M_i$

$$\begin{tikzcd} SN(L_1) \arrow[r] \arrow[d] & SN(L_2) \arrow[d] \\ L_1 \arrow[r, "\phi"] & L_2 \end{tikzcd}$$

$\Rightarrow \phi$ extends to symplectomorphism

$$\underset{W_1}{\underset{\cap}{Op(L_1)}} \longrightarrow \underset{W_2}{\underset{\cap}{Op(L_2)}} \qquad \begin{pmatrix} M_1 \to M_2 \\ V_1 \to V_2 \end{pmatrix}$$

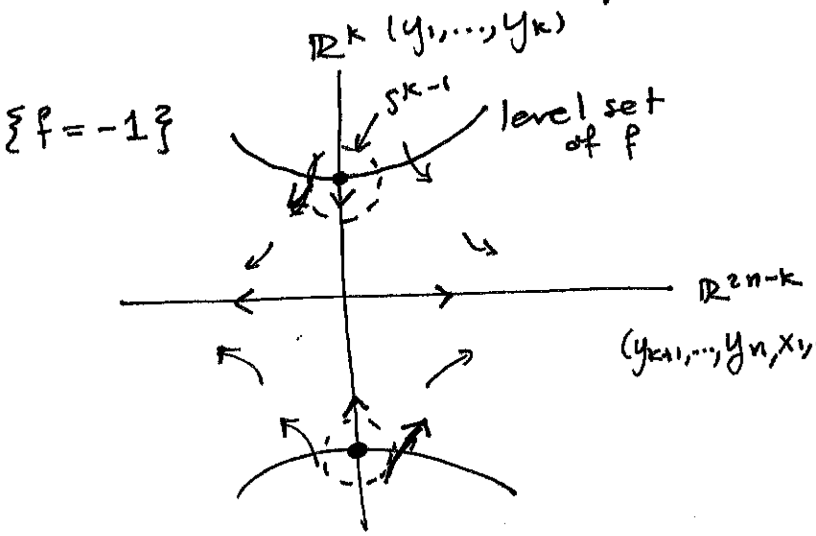


Symplectic handle

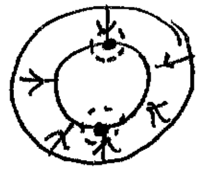
(Weinstein)

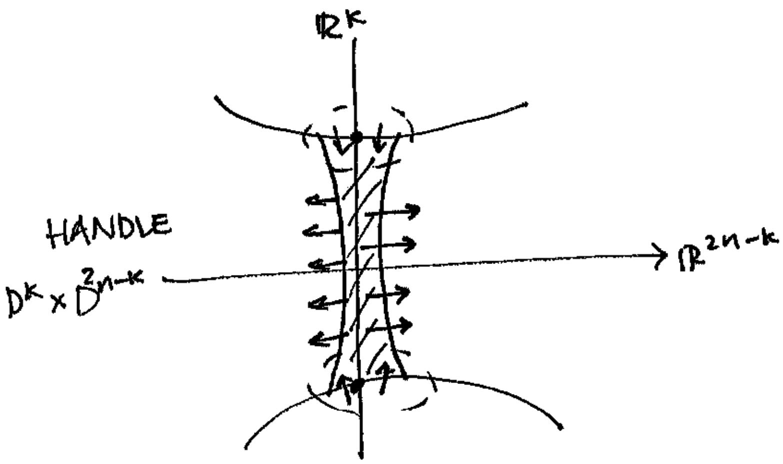
$(\mathbb{R}^{2n}, \ \omega = \sum dx_i \wedge dy_i)$

$$f = \sum_{i=1}^{k} \left(x_i^2 - \tfrac{1}{2} y_i^2\right) + \tfrac{1}{4} \sum_{i=k+1}^{n} \left(x_i^2 + y_i^2\right)$$

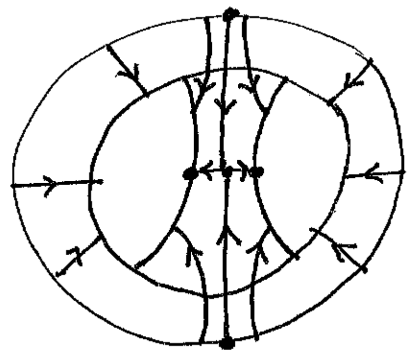


Now, ~~M×I~~ we need to construct the rest of the handle





Now we have a contact structure.



$$V = \sum_{i=1}^{k} 2x_i \partial_{x_i} - y_i \partial_{y_i} + \frac{1}{2} \sum_{i=k+1}^{n} x_i \partial_{x_i} + y_i \partial_{y_i}$$

Let's check that it is a Liouville vector field:

$$\omega(V, -) = \sum_{i=1}^{k} 2x_i dy_i + y_i dx_i + \sum_{i=}^{n} x_i dy_i - y_i dx_i$$

$$d(\omega(V,-)) = \underbrace{\sum_{i=1}^{k} 2 dx_i \wedge dy_i + dy_i \wedge dx_i + \sum_{i=k+1}^{n} dx_i \wedge dy_i - dy_i \wedge dx_i}_{- \omega}$$