François-Simon: Weinstein handles and contact surgery

Smooth surgery:

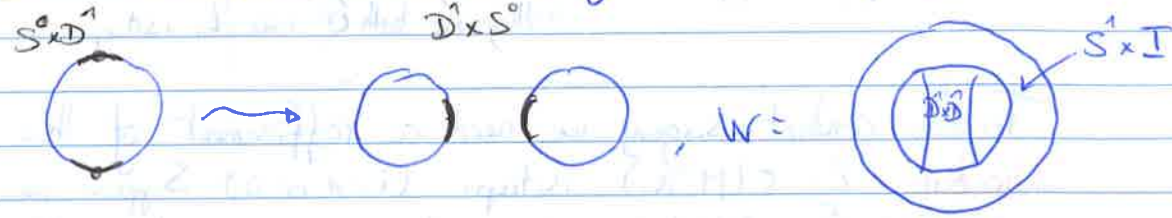
M^n manifold; consider $S^k \times D^{n-k} \subseteq M$; think of it as $S^k \subseteq M$ together with a trivialization of its normal bundle.

Observe that $\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1} = \partial(D^{k+1} \times S^{n-k-1})$.

$\rightsquigarrow$ get $M' = (M \setminus \mathrm{int}(S^k \times D^{n-k})) \cup (D^{k+1} \times S^{n-k-1})$

Fact: M and M' are cobordant, by $W = M \times I \cup D^{k+1} \times D^{n-k}$.

ex:



Contact surgery:

$(M^{2n+1}, \xi = \ker \alpha)$, let $S^k \subseteq M$ isotropic, with trivialization of the "conformal symplectic normal bundle". We'd like to have M' and W as above, but in a contact/symplectic way.

$\to$ Isotropic submanifolds: $L \subseteq (M, \xi)$ isotropic if $TL \subseteq \xi$.

$\xi = \ker \alpha \Rightarrow d\alpha|_\xi$ is a symplectic form on ξ

$\xi = \ker(f\alpha) \Rightarrow d(f\alpha)|_\xi = f \, d\alpha|_\xi$, so get a conformal symplectic structure.

If $\eta \subseteq \xi$, then $\eta^\perp = \{x \in \xi \mid d\alpha(x,v) = 0 \ \forall v \in \eta\} \subseteq \xi$ as well.

In our case: L isotropic $\Rightarrow TL \subseteq TL^\perp$.

The conformal symplectic normal bundle: $CSN(L) = TL^\perp / TL$; it has a conformal symplectic structure induced by ξ.

Theorem: let $L_i \subseteq (M_i, \xi_i)$ isotropic ($i = 1$ or 2). Suppose we have

$$\begin{array}{ccc} CSN(L_1) & \xrightarrow{\Phi} & CSN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\varphi} & L_2 \end{array}$$

where Φ is an isomorphism of conformal symplectic structures, and φ is a diffeo.

Then: φ extends to a contactomorphism $\mathcal{O}p(L_1) \to \mathcal{O}p(L_2)$.

Proof: $L \subseteq (M, \xi)$ isotropic: $TL \subseteq TL^\perp \subseteq \xi|_L \subseteq TM|_L$. We have

$$\underset{\substack{\shortparallel \\ TM|_L / TL}}{NL} \cong TM|_L / \xi|_L \oplus \xi|_L / TL^\perp \oplus TL^\perp / TL$$

$$\cong \langle R_\alpha \rangle \oplus T^*L \oplus CSN(L)$$

The identification $\xi|_L / TL^\perp \cong T^*L$ comes from $\xi|_L \to T^*L : Y \mapsto d\alpha(Y, -)$;

by non-degeneracy of $d\alpha$, it is surjective, and the kernel is by definition $TL^{\perp}$.

Get $\psi : NL_1 \xrightarrow{\sim} NL_2$: $\begin{cases} R_{\alpha_1} \mapsto R_{\alpha_2} \\ (\phi^*)^{-1} : T^*L_1 \to T^*L_2 \\ \Phi : CSN(L_1) \to CSN(L_2) \end{cases}$

$\Rightarrow$ By tubular neighborhood theorem, get $\tilde{\phi} : Op(L_1) \to Op(L_2)$, such that $D\tilde{\phi}|_{L_1} = D\phi \oplus \psi$, and $\phi^*\alpha_2 = \alpha_1$ on L_1. Make the contact forms agree everywhere by Gray's theorem.

they are both 0, since L_i isotropic.

$\square$

To do contact surgery, we need a raffinement of this:

Theorem. $L_i \subseteq (M, \alpha_i)$ isotropic ($i = 1$ or 2). Suppose we have

$$\begin{array}{ccc} SN(L_1) & \xrightarrow{\Phi} & SN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

where Φ is an isomorphism of symplectic bundles, and ϕ is a diffeo.

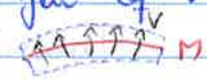
Then, ϕ extends to $\tilde{\phi} : Op(L_1) \to Op(L_2)$ st $\tilde{\phi}^*\alpha_2 = \alpha_1$.

Rem: so, here, we fix the forms.

$\rightarrow$ Liouville vector fields: vector field V on (W, ω) such that $\mathcal{L}_V\omega = \omega$ ($\Leftrightarrow d\lambda = \omega$, where $\lambda = \omega(V, -)$).

ex: for $(M, \xi = \ker(\alpha))$, we have the symplectization $(\mathbb{R}_t \times M, d(e^t\alpha))$; then $\partial/\partial t$ is Liouville.

ex: for $M^{2n-1} \subseteq (W, \omega)$ and $V \pitchfork M$ Liouville, we say M is "contact type"; indeed, $\alpha = \lambda|_M$ is a contact form on M, and the flow of V gives an identification $Op(M) \overset{\text{symp}}{\cong} ((-\varepsilon, \varepsilon) \times M, d(e^t\alpha))$.



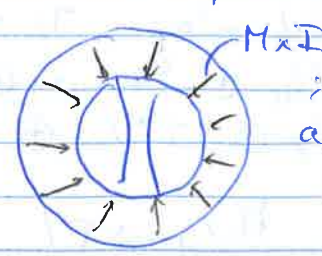
Theorem. $L_i \overset{\text{isotropic}}{\subseteq} M_i^{2n-1} \subseteq (W_i^{2n}, \omega_i)$ ($i = 1$ or 2); $V_i \pitchfork M_i$ Liouville.

If we have

$$\begin{array}{ccc} SN(L_1) & \longrightarrow & SN(L_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

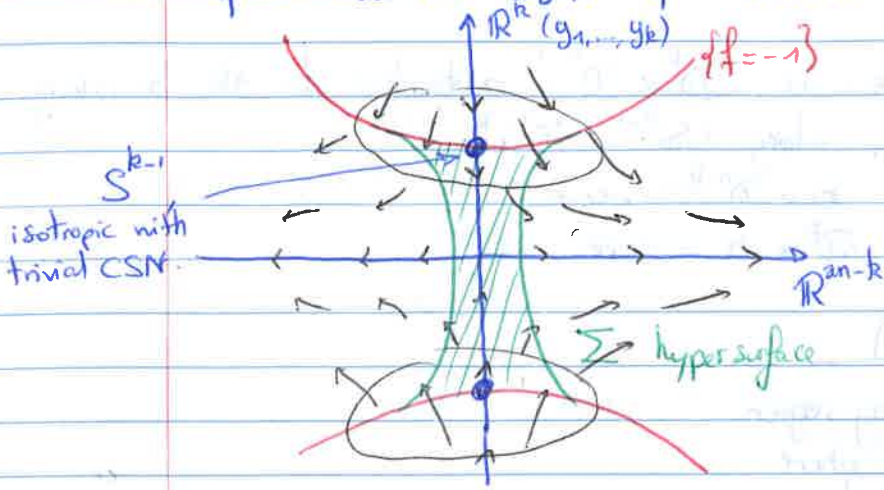
then ϕ extends to a symplectomorphism $Op_{W_1}(L_1) \to Op_{W_2}(L_2)$, such that it identifies M_1 with M_2 and V_1 with V_2.

So for contact surgery:



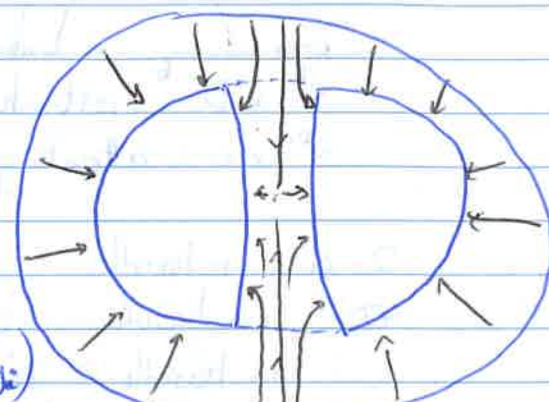
; now we need to build a symplectic handle ("Weinstein" handle).

→ Symplectic (Weinstein) handles: in $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$, consider $f = \sum_{i=1}^{k} (x_i^2 - \frac{1}{2} y_i^2) + \frac{1}{4} \sum_{i=k+1}^{n} (x_i^2 + y_i^2)$, then $V = \nabla f$ is Liouville.



The black circles represent the attaching region of the handle; Σ is a hypersurface such that $\Sigma \pitchfork V$, and the region inside it is the handle.

After gluing that handle, we get:



Rem: $V = \sum_{i=1}^{k} (2x_i \partial x_i - y_i \partial y_i) + \frac{1}{2} \sum_{i=k+1}^{n} (x_i \partial x_i + y_i \partial y_i)$

Just so we do it at least once, let's check it's Liouville:
$\omega(V, -) = \sum_{i=1}^{k} (2x_i dy_i + y_i dx_i) + \frac{1}{2} \sum_{i=k+1}^{n} (x_i dy_i - y_i dx_i)$
$\Rightarrow d(\omega(V,-)) = \sum_{i=1}^{k} (2dx_i \wedge dy_i + dy_i \wedge dx_i) + \frac{1}{2} \sum_{i=k+1}^{n} (dx_i \wedge dy_i - dy_i \wedge dx_i) = \sum_{i=1}^{n} dx_i \wedge dy_i = \omega.$

Idea: can have at most half the directions pointing inwards (the y_i's), because they need to be compensated by the other ones.

Rem: really what's going on, is that in the half-dimension, we have the holomorphic function z^2; the level sets of the real part and the imaginary part are transverse, by holomorphicity.

Rem: if $k = n$, $SNL = 0$, so we don't have to worry about SNL_1 and SNL_2 being ~~isomorphic~~ symplectomorphic.