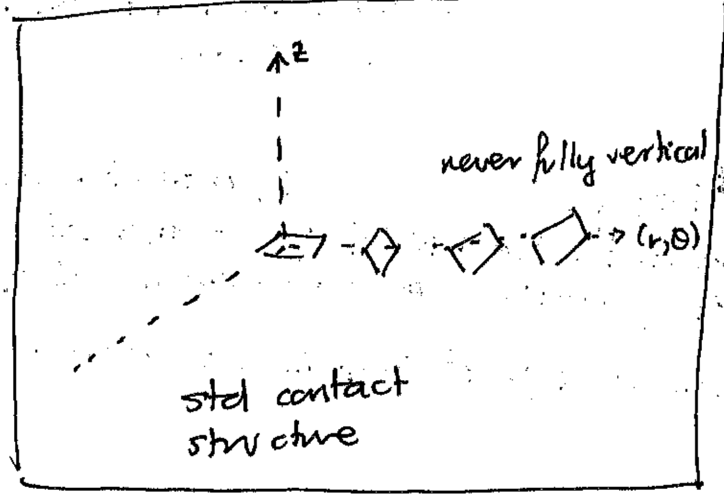
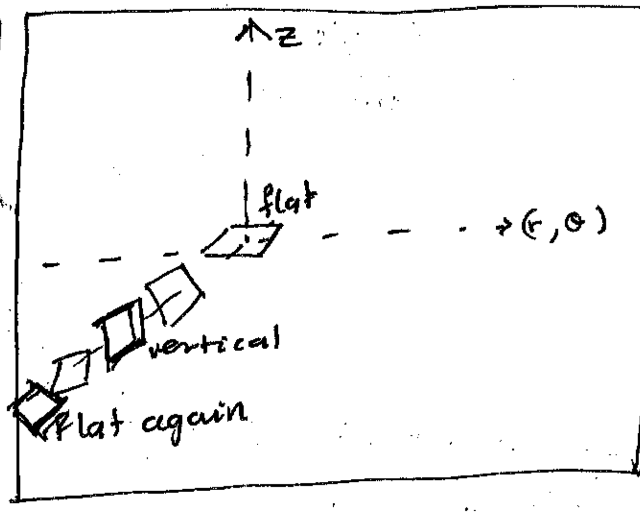
# INTRODUCTION

## Symplectic Fillings

From contact viewpoint: 1982 Bennequin proves that $(\mathbb{R}^3, dz - y\,dx)$ is not contactomorphic to $(\mathbb{R}^3, \cos r\,dz + r\sin r\,d\theta)$

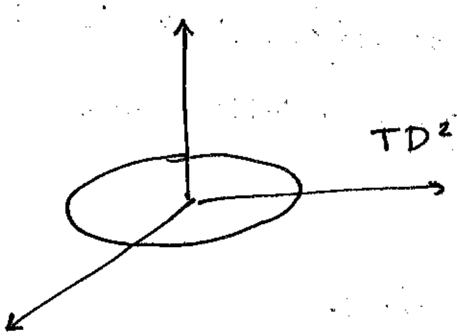



Newer viewpoint: (Gromov 85) shows that the compactifications $(S^3, \xi_0)$ and $(S^3, \xi_{OT})$ are different because $\xi_0$ bounds and $\xi_{OT}$ doesn't.

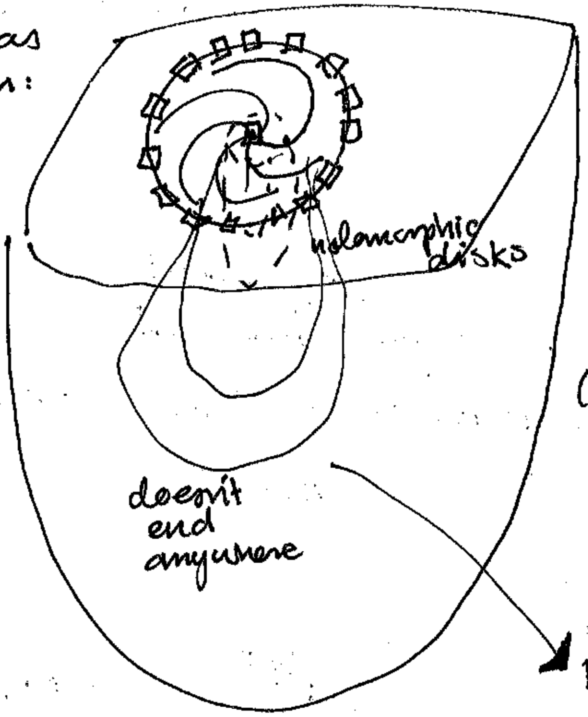
SKETCH OF PROOF:

- $(S^3, \xi_0)$ is bounds Darboux ball $(D^4, \omega_\alpha)$
- Suppose that $(S^3, \xi_{OT})$ bounds some symplectic mfld (which we can assume is nice e.g. $(W, \lambda)$ where $d\lambda = \omega$).

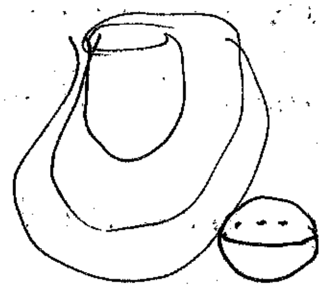


$TD^2 \cap \xi_{OT}$ gives a line field w/ singularities where they coincide. (here is boundary + origin)

• Disk has foliation:




bubble

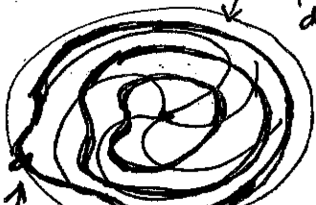


DOESNT EXIST BY EXACTNESS of $(\omega, \lambda)$

Bishop 1-parameter fam
Explicit local model of holomorphic disks, which can never touch $\partial D^2$.

Remk: If you get rid of exactness still find that $\xi_{OT}$ is not weakly fillable.

boundaries of holomorphic disks



this can't happen!

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Remark: Not every $(2n+1)$ manifold has a smooth filling

E.g $\frac{SU_3}{SO(3)}$, $\mathbb{C}P^2 \times_{conj} S^1$

But $\Omega^u_{2n+1} = 0$, every manifold $Y^{2n+1}$ which is almost contact is the boundary of a manifold $W^{2n+2}$ with $W$ almost complex.

$\Omega^U_{2n+1}$ = unitary bordism ring.

**Symplectic viewpoint** Distinguish $W_1$ & $W_2$ via their boundaries which have same contact structure.

Consider the singular complex mfld of $\dim_{\mathbb{R}}=6$

$$V = \{ z_0^7 + z_1^2 + z_2^2 + z_3^2 = 0 \} \subset \mathbb{C}^4$$

$\Rightarrow$ look at $\mathbb{C}^4 \setminus V$ & modify its topology

$\Rightarrow$ [McLean 08] $\widetilde{\mathbb{C}^4 \setminus V} \underset{\text{DIFF}}{\cong} \mathbb{R}^8$

Q: Is it symplectomorphic to $(\mathbb{R}^8, \omega_0)$?

Then, by looking at the boundaries ($SH^+$) you distinguish exact symplectic type; but this gives $\omega$-invariant.

Q1: What about contact structures with NO symplectic fillings? Yes they exist: SW are a good obstruction for 3-mflds

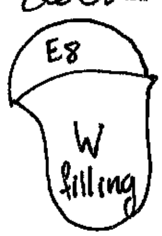
Glimpse:

$$\Sigma(2,3,5) = \{ z^2 + x^3 + y^5 = 0 \} \cap S^5 \subseteq \mathbb{C}^3$$

is the Poincare homology sphere

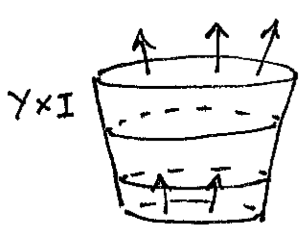
In fact $\partial E_8 = \Sigma_{2,3,5}$ & this one does not have a symplectic filling.

But $\Sigma(2,3,5)$ does.




positive scalar curvature $\Rightarrow b_2^+(W) = 0$

smooth topology tells you that such a 4-mfld cannot exist $\square$




CAPS VS FILLINGS

$\uparrow$ $\omega$ expanding

Caps 

$(S^3, \xi_{OT})$ has a cap. but it doesn't have a filling.

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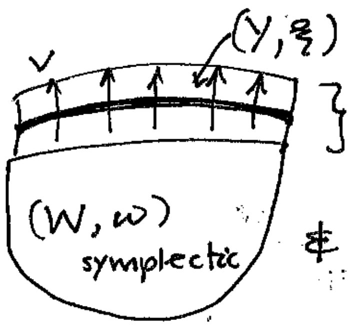
## ★ TYPES OF FILLINGS ★

~~STEIN~~

STRONG



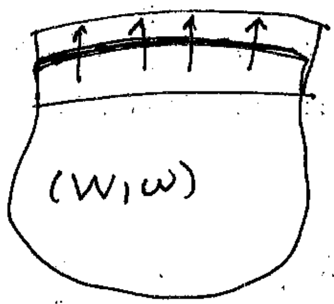

$\} \ \omega = d\lambda$

& exact near boundary

v must be upper transverse

~~Exercise show~~

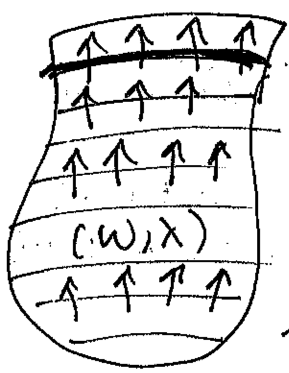
EXACT




$d\lambda = \omega$ on all of $W$

STEIN — an exact filling that is well behaved wrt Morse function

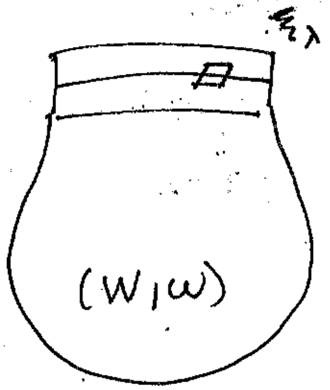
(Weinstein)




$\xrightarrow{\phi} \mathbb{R}$

level sets = contact mflds

weak (3d)




here you choose contact structure & then compare to symplectic structure.

No primitive, nor induced $\xi$.

does not allow for concatenation.

want
$\omega_x(\xi_x) > 0$

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STEIN $\subseteq$ EXACT $\subseteq$ STRONG $\subseteq$ WEAK

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Q2: Are there contact manifolds with infinitely many fillings?

e.g (stein). $\boxed{\text{Yes}}$ we will have $(Y^3, \xi)$ admitting a family of stein fillings $X_n^4$, with different homology groups. $H_0(X_n) = \mathbb{Z} \oplus \mathbb{Z}_m$.

- Sketch: try to factor elements of the mapping class group $\Gamma_{g,h}^k$ of a surface into two positive different ways.

Day 1: Lefschetz fibration + open book.

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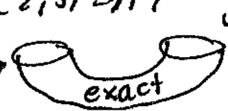
The inclusions are all strict

EXAMPLES in 3 dim:

$(T^3, \xi_n)$
$\alpha_n = \cos(nx)dy + \sin(nx)dz$

stein $\subseteq$ exact $\subseteq$ strong $\subseteq$ weak

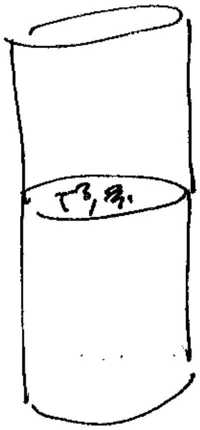
$\exists$ contact structures
$(\Sigma(2,3,5), \xi_1)$ ③
$(\Sigma(2,3,5), \xi_0)$




② Start w/ weak filling of a homology sphere
$\Rightarrow$ its a strong filling ~~but not~~ by cohomological conditions.

① THM 1: $\xi_n$ are weakly fillable
(by $T^2 \times D^2$)
(2) Only $\xi_1 = \partial(T^* T^2)$. is strongly fillable

① continued




$\Leftarrow$ somenthing $= \mathbb{R}^4$ at $\infty$.

n fold over $\rightarrow$


Take lagrangian $T^2$

$T^2_{clifford}$

$$\mathcal{O}_p(T^2_{clifford}) \subseteq (\mathbb{R}^4, \omega_0)$$
$$\| $$
$$T^*T^2$$

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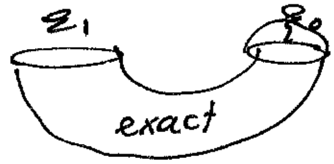
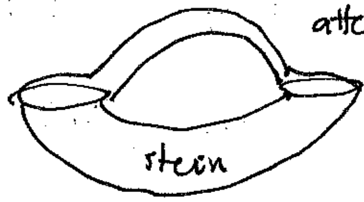
③ continued




$\leftarrow$ can be capped

why isn't it stein?




attached a handle

Wanted to use:

THM: Given a stein filling of $(Y_1^3, \xi_1) \# (Y_2^3, \xi_2)$

$\Rightarrow (Y_i^3, \xi_i)$ is stein fillable.