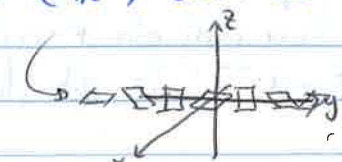
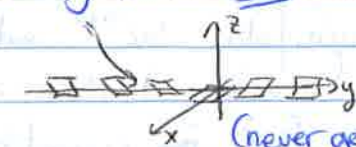
Roger, Laura - Introduction, types of fillings / fillability

From contact viewpoint:

[Bennequin, 1982]: proves that $(\mathbb{R}^3, dz - y\,dx)$ is not contactomorphic to $(\mathbb{R}^3, \cos r\, dz + r \sin r\, d\theta)$. $\cong (\mathbb{R}^3, dz + r^2 d\theta)$


(keeps turning)


(never gets vertical)

Rem: $\exists$ diffeo of $\mathbb{R}^3$, taking $dz - y\,dx$ and making it twist in one direction. But $\cos r\, dz + r \sin r\, d\theta$ twists in every direction (ie value of θ).

Newer perspective:

[Gromov 1985]: shows that the compactifications (S^3, ξ_0) and (S^3, ξ_{OT}) differ: the 1^{st} one "bounds" a symplectic manifold, but not the 2^{nd} one.

Proof: (S^3, ξ_0) bounds a Darboux (D^4, ω_0). Let's prove that (S^3, ξ_{OT}) does not bound. Suppose it does: it bounds $(W, \omega = d\lambda)$ (take ω exact to be nice). The disk  has a characteristic foliation, $\{z=0, r \leq \pi\}$ looking like . [Bishop]


(S^3, ξ_{OT})
$(W, d\lambda)$

Start filling by holomorphic disks in 1-param family. Gromov proves that the family persists, until compactness kicks off:

$\rightarrow$ Could have bubble (green), but does not exist by Stokes because $(W, d\lambda)$ is exact.

$\rightarrow$ ∂ of disks can ~~not~~ be tangent to leaves of characteristic foliation of OT disk by compactness, but this can't happen by the maximum principle.

So one of these 2 must happen, but none can $\Rightarrow$ Contradiction. $\square$

Picture for that local family:  D^3 (1^{st} one is constant)

There is an explicit local model in $\mathbb{C}^2$, for that family of disks. The ∂ of the disks in that family are in D^2_{OT}; by maximum principle, it can not be tangent to ∂D^2_{OT}. These disks are all disjoint.

No! 

Rem: not every $(2n+1)$-manifold has a filling, like $\mathbb{CP}^2$ in even dim case. In odd dim case: $SU_3/SO(3)$, $\mathbb{CP}^2 \times_{conj} S^1$.
But: $\Omega^U_{2n+1} = 0$, i.e. every (Y^{2n+1}, ξ) contact is the boundary of a manifold W^{2n+2} with W almost complex.
↳ unitary bordism ring ↳ or even almost contact
⚠ so far, compatibility between ξ and J is not clear.

From symplectic viewpoint:
We want to distinguish W_1 and W_2 via their contact boundaries.
ex: $V^6 := \{ z_0^7 + z_1^2 + z_2^2 + z_3^2 = 0 \} \subseteq \mathbb{C}^4$; consider $\mathbb{C}^4 \setminus V$, then modify its topology a bit (handle attachment); call the new one $\widetilde{\mathbb{C}^4 \setminus V}$.
[McLean 2008]: $\widetilde{\mathbb{C}^4 \setminus V} \overset{\text{diffeo}}{\cong} \mathbb{R}^8$, is it symplectomorphic to $(\mathbb{R}^8, \omega_0)$?
By looking at the boundaries (with SH^+) (ie contact structure at ∞), you can distinguish them. By connect-summing copies of $\widetilde{\mathbb{C}^4 \setminus V}$, get ∞ly many of those, all not symplectomorphic.
Actually: we distinguish the exact symplectomorphism type.

Q1: what about contact structures with no filling? (Day 3/4)
A1: yes, they exist: Seiberg-Witten invariants are a good obstruction for 3-manifolds.
ex: $\Sigma(2,3,5) = \{ z^2 + x^3 + y^5 = 0 \} \cap S^5 \subseteq \mathbb{C}^3$ is the Poincaré homology sphere.
In fact, $\partial E_8 = -\Sigma(2,3,5)$ ($E_8 = E_8$-plumbing), $\boxed{\text{Smoothly}}$.
Seiberg-Witten: ∂ has positive scalar curvature $\Rightarrow b_2^+(\text{filling}) = 0$.

By smooth topology (Donaldson's diagonalization), X^4 can not exist.

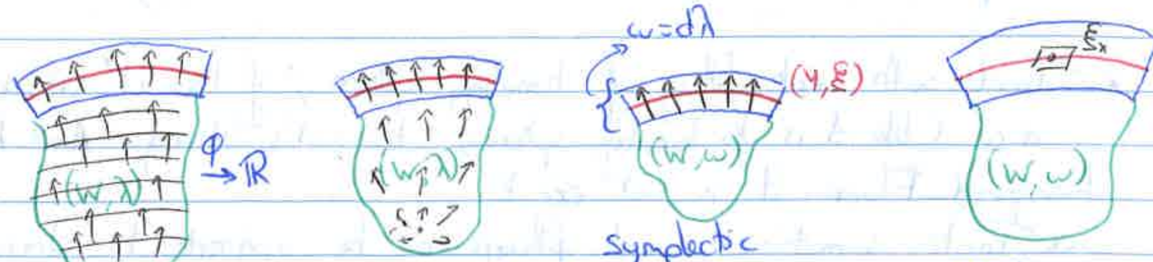
Strange phenomenon: can always find caps:  (Y, ξ). Also, there is a maximum principle, but no minimum principle.

Now, we'll see the different types of fillings.

(Weinstein)

Stein $\subseteq$ Exact $\subseteq$ Strong $\subseteq$ Weak (3D)

(black arrows = dual to λ)



can find $\varphi: W \to \mathbb{R}$ st V is ∇-like. $\quad$ $\omega = d\lambda$ everywhere $\quad$ + exact near ∂ $\quad$ $\omega_x(\xi_x) > 0$, $\forall x$, and orientations should agree.

Stein, exact & strong induce a contact structure on the boundary; for weak, we are comparing ω and ξ.
Rem: for Stein, exact, strong: we can concatenate fillings; this is not the case for weak fillings

⚠ distinction between concave and convex ends in subtle, you can not just switch the sign of the forms.

Q2: are there contact manifolds with infinitely many Stein fillings?
Day 2: yes; we will have (Y^3, ξ) admitting a family X_n^4 of Stein fillings, with $H_*(X_n) = \mathbb{Z} \oplus \mathbb{Z}_n$ ($\Rightarrow$ different).
Sketch: try to factor elements of $\Gamma_{g,n}^k$ (mapping class group of genus g surface) in 2 positive different ways. This uses (day 1) Lefschetz fibrations and open books. This is very algebraic; the only point using Stein-ness is the word "positive" about.

Rem: the inclusions above are all strict! ~~(in 3D)~~
Examples in 3D:

Strong ⊊ Weak

* (T^3, ξ_n) with $\alpha_n = \cos(nx)dy + \sin(nx)dz$.

Theorem: (1) ξ_n are all weakly fillable (by $T^2 \times D^2 = T^*T^2$).
(2) Only $\xi_1 = \partial(T^*T^2)$ is strongly fillable.

Proof: (1) is easy. For (2): take Lagrangian $T^2_{\text{Clifford}} \subseteq (\mathbb{R}^4, \omega_0)$; a neighborhood of this is $\simeq T^*T^2$. Consider the complement, and take an n-fold cover; it has n ends (because the complement had 1 end).

⑥

So it looks like $\mathbb{R}^4$ at infinity, hence by Gromov there is only one possibility: $n=1$. So we can not ~~weak~~ strongly fill it if $n \neq 1$.

Exact & strong

* Start with weak filling of homology sphere; if the H^* of boundary is nice (like it is for homology spheres), then it's strong. And by Heegaard-Floer: it is not exact.

$\rightsquigarrow$ Trick: sometimes, weak fillings can be upgraded to strong fillings, if the homology of the boundary is nice enough.

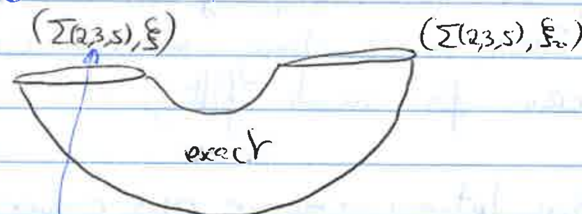
Stein & exact

* [McDuff]: constructs exact filling with 2 convex boundaries

$\Rightarrow$ there must be a 3-handle

$\Rightarrow$ this can not be Stein.

Other example:



The exact filling is explicit (convex combination of ξ and ξ_0). Why can we not Stein-fill it? Attach a 1-handle. If $(\Sigma(2,3,5), \xi)$ has a Stein filling, get a contradiction from:

Theorem [Eliashberg] if we have a Stein-filling of $(Y_1^3, \xi_1) \# (Y_2^3, \xi_2)$, then both (Y_i^3, ξ_i) are Stein fillable.

Rem. we have no clue as to what happens in high-dimensions.

ex: $\mathbb{CP}^5$ does not have Stein filling by homotopical reasons, but we don't know if it has an exact filling.

ex. Stein fillings of exotic structures on S^5 are related to hard questions in other areas (s.t. geometric group theory).